PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME VIII.

SESSION 1889-90.

EDINBURGH:

1890.

PRINTED BY

JOHN LINDSAY, HIGH STREET, EDINBURGH.

INDEX.

Note.—The Committee desire to express their indebtedness to T. B. SPRAGUE, Esq., for a liberal donation, which has enabled them to print the figures to Professor Steggall's paper.

PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

EIGHTH SESSION, 1889-90.

First Meeting, 8th November 1889.

George A. Gibson, Esq., M.A., President, in the Chair.

For this Session the following Office-Bearers were elected :—

President—Mr A. Y. Fraser, M.A., F.R.S.E.

Vice-President—Mr R. E. Allardice, M.A., F.R.S.E.

Secretary—Mr John Alison, M.A., F.R.S.E.

Treasurer—Rev. John Wilson, M.A., F.R.S.E.

Editor of Proceedings—Mr R. E. Allardice.

Committee—

Professor Steggall, M.A., F.R.S.E.; Messrs J. W. Butters, A. C. Elliott, D.Sc., C.E.; A. C. Mitchell, B.Sc., F.R.S.E.; J. T. Morrison, M.A., B.Sc.; and William Wallace, M.A.

Green's and allied theorems: a historical sketch.

By GEORGE A. GIBSON, M.A.

[ABSTRACT.]

The chief purpose of the paper was to indicate the rise of transformations of the type $\int\int\int\frac{dV}{dx}dxdydz=\int V\cos\alpha dS$ where the integral in the first member of the equation is taken throughout a closed surface, and that in the second member over the surface, α being the angle made with the axis of x by the normal to the element dS drawn outwards. It is on this transformation the analytical proof of Green's theorem depends, and it was shown to have been employed in various forms by Poisson, Duhamel, Gauss, and others, before Green's essay was generally known on the Continent. It may be observed that the essay was published at Nottingham in 1828, and seems to have been unknown to continental mathematicians till its reprint in *Crelle's Journal*, vols. 39 (1850), 44 (1852), and 47 (1854). The following references were given in the paper:—

Lagrange, in the *Mécanique Analytique* (2nd edition, 1811), Part I., sect. vii., arts. 29, 30, gives the transformation:—

$$\mathbf{S}\lambda'(\delta x'.dydz+\delta y'.dzdx+\delta z'.dxdy)=\mathbf{S}\lambda'(\cos\alpha'.\delta x'+\cos\beta'.\delta y'+$$

$\cos\gamma'.\delta z')ds^2$, ds^2 being an element of surface.

Laplace, in the *Supplément à la Théorie de l'Action Capillaire* (which forms a supplement to Liv. x., Part II., of the *Mécanique Céleste*, published 1806)—*Oeuvres Complétes*, 1880, t4. pp. 428–432 —transforms the integral

$$\int\int dxdy\left(\frac{d}{dx}\cdot\frac{p}{\sqrt{1+p^2+q^2}}+\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}}\right)$$

taken over the area of a section of a cylinder, whose generators are parallel to the axis of z, into the integral $\pm\int\frac{pdy-qdx}{R}$ taken along the boundary of the section, the $+$ sign holding for the part of the curve convex to the axis of x, the $-$ sign for that concave to the same axis.

Gauss gives a series of remarkable theorems, closely related to the transformation in question, in the introductory articles of his

"Theoria Attractionis corporum sphæroidicorum ellipticorum homogeneorum," *Comment. Soc. Reg. Gott.*, vol. ii. (1813), Werke, Bd. V. Great care is taken to make the proofs quite general. Some of these theorems will be found in Williamson's *Integral Calculus* (3rd edition), arts. 192, 193. Transformations of a like character occur in Gauss's Memoirs, "Principia Generalia Theoriæ Figuræ Fluidorum in Statu Aequilibrii," *Comment. Soc. Reg. Gott.*, vol. vii. (1830), Werke, Bd. V., and "Allgemeine Lehrsätze in Bez. auf die . . . Anziehungs-und Abstossungs-Kräfte," *Resultate des Magn. Vereins*, 1840, Werke, Bd. V. The proof of Poisson's equation, in §§ 9, 10 of the latter memoir, depends really on the transformation of a volume integral of the kind mentioned into the difference of a volume and of a surface integral, the manner of carrying out the transformation being quite special.

Poisson in various memoirs uses with considerable effect the transformation employed (later) by Green, and establishes in his earlier memoirs particular cases of Green's theorem. In his "Mémoire sur la Théorie du Magnetisme," *Mém. de l'Inst.*, t.v., 1826, read February 2, 1824, pp. 294-298, he proves the equation

$$\iiint k'\left(\alpha'\frac{d}{dx'}.\frac{1}{\rho}+\beta'\frac{d}{dy'}.\frac{1}{\rho}+\gamma'\frac{d}{dz'}\frac{1}{\rho}\right)dx'dy'dz'$$
$$=\iint k'(\alpha'\cos l'+\beta'\cos m'+\gamma'\cos n')\frac{d\omega'}{\rho}$$
$$-\iiint\left(\frac{d(\alpha'k')}{dx'}+\frac{d(\beta'k')}{dy'}+\frac{d(\gamma'k')}{dz'}\right)\frac{dx'dy'dz'}{\rho}$$

where $\rho^2=(x-x')^2+(y-y')^2+(z-z')^2$ and the volume integrals are taken throughout a closed surface and the double integrals over the surface. He considers the possibility of a line cutting the surface in more points than two, but makes no reference to Gauss's memoir of 1813. In § 18 of the same memoir he discusses the case in which $1/\rho$ becomes infinite within the limits of integration.

Another important example from Poisson is furnished by § 89 of the sixth chapter of his *Théorie de la Chaleur* (Paris, 1835; a date later than that of the publication of Green's essay and of the memoir of Duhamel referred to below). He arrives at the following equation :—

$$\frac{d}{dt}.\iiint cu\text{P}dxdydz$$

$$= \iiint \mathrm{P}\left\{\frac{d}{dx}\left(k\frac{du}{dx}\right)+\frac{d}{dy}\left(k\frac{du}{dy}\right)+\frac{d}{dz}\left(k\frac{du}{dz}\right)\right\}dxdydz$$

$$= \iint \mathrm{P}k\left(\frac{du}{dx}\cos\alpha+\frac{du}{dy}\cos\beta+\frac{du}{dz}\cos\gamma\right)d\mathrm{S}$$

$$- \iint uk\left(\frac{d\mathrm{P}}{dx}\cos\alpha+\frac{d\mathrm{P}}{dy}\cos\beta+\frac{d\mathrm{P}}{dz}\cos\gamma\right)d\mathrm{S}$$

$$+ \iiint u\left\{\frac{d}{dx}\left(k\frac{d\mathrm{P}}{dx}\right)+\frac{d}{dy}\left(k\frac{d\mathrm{P}}{dy}\right)+\frac{d}{dz}\left(k\frac{d\mathrm{P}}{dz}\right)\right\}dxdydz$$

The last equation is in form the extension of Green's theorem given in Thomson and Tait's *Natural Philosophy*, vol. i., p. 168.

Another interesting example from Poisson occurs in a "Mémoire sur l'Équilibre des Fluides," *Mém. de l'Inst.*, t.ix, 1830, read Nov. 24, 1828, pp. 22-24.

Ostrogradsky, in a memoir with the title "Note sur la Théorie de la Chaleur," *Mém. de l'Acad. de St Petersbourg*, 6s. t.i (1831), read Nov. 5, 1828, p. 129, establishes the theorem

$$\int\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right)\omega=\int\left(\mathrm{P}\cos\lambda+\mathrm{Q}\cos\mu+\mathrm{R}\cos\nu\right)s$$

where ω is a volume-element, s a surface-element, and P, Q, R the values of p, q, r at s.

Duhamel, in Note 1. appended to a memoir on the "Theory of Heat," published in the *Journ. Ecol. Polyt.*, t.xiv, cah. 22 (1833), gives, at pp. 67-71, a proof of the equation

$$\int \mathrm{U}'\left(\frac{d^2\mathrm{U}}{dx^2}+\frac{d^2\mathrm{U}}{dy^2}+\frac{d^2\mathrm{U}}{dz^2}\right)dxdydz$$

$$=\int \mathrm{U}\left(\frac{d^2\mathrm{U}'}{dx^2}+\frac{d^2\mathrm{U}'}{dy^2}+\frac{d^2\mathrm{U}'}{dz^2}\right)dxdydz$$

$$+\int \mathrm{U}\left(l\frac{d\mathrm{U}'}{dx}+m\frac{d\mathrm{U}}{dy}+n\frac{d\mathrm{U}}{dz}\right)d\mathrm{S}$$

$$-\int \mathrm{U}\left(l\frac{d\mathrm{U}'}{dx}+m\frac{d\mathrm{U}'}{dy}+n\frac{d\mathrm{U}'}{dz}\right)d\mathrm{S}$$

l, m, n, being the direction cosines of the outward normal at dS.

Lamé, in the same volume, p. 204 and p. 231, gives investigations of a similar character.

Sir W. Thomson, in Article XII. of his "Electrostatics and Magnetism" (reprinted from the *Camb. Math. Jour.*, Nov., 1842, and Feb. 1843), furnishes examples of the analysis with which the

paper dealt, and which, he says, was suggested to him by the analysis used by Poisson in the article of his *Théorie de la Chaleur*, quoted above.

Questions of priority are usually somewhat difficult to answer; but while it seems clear that the theorem generally quoted as Green's was given independently of Green, yet the importance which he rightly attached to it, and the splendid use to which he put it, amply justify us in keeping to the customary mode of citation.

Some new Properties of the Triangle.

By J. S. MACKAY, M.A., LL.D.

[The substance of this paper will be included in Dr Mackay's paper on the triangle in the first volume of the *Proceedings*, now about to be published.]

Second Meeting, 13th December 1889.

R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

A special case of three-bar motion.

By Professor STEGGALL.

The questions involved in the consideration of three-bar motion have attracted a good deal of attention (*Proceedings of Mathematical Society of London* passim, and elsewhere); but I am not aware of any complete account of the figures that can be derived from such a motion. The present paper gives a complete list of all the different kinds of curve that are obtained by a tracing point at the middle of the middle bar, the two outer bars being equal.

It may be advisable to briefly obtain the general equation to the curve traced by any point on the middle bar, without any condition of equality in the lengths of the other two.

Let $2a$ be the distance of the fixed centres, b, $2c$, d the lengths of the three bars in order, h the distance of the tracing point from the middle of the middle bar measured from the bar b, θ, ϕ, ψ the

angles which the bars in order make with the line joining the fixed centres: take this line as axis of x, and a perpendicular through its middle point as that of y.

We have at once, on reference to a diagram,

$$\begin{aligned} b\cos\theta + (h+c)\cos\phi &= a + x, \\ b\sin\theta + (h+c)\sin\phi &= \quad y, \\ d\cos\psi + (h-c)\cos\phi &= x - a, \\ d\sin\psi + (h-c)\sin\phi &= \quad y. \end{aligned}$$

Whence

$$\begin{aligned} b^2 &= y^2 + (a+x)^2 + (h+c)^2 \\ &\quad - 2(h+c)\{(x+a)\cos\phi + y\sin\phi\}, \\ d^2 &= y^2 + (x-a)^2 + (h-c)^2 \\ &\quad - 2(h-c)\{(x-a)\cos\phi + y\sin\phi\}. \end{aligned}$$

Whence

$$4\{(hx+ac)\cos\phi + hy\sin\phi\} = 2(x^2+y^2+a^2+h^2+c^2) - b^2 - d^2,$$
$$4\{(cx+ah)\cos\phi + cy\sin\phi\} = 4ax + 4hc - b^2 + d^2.$$

Now the eliminant of

$$\begin{aligned} P\cos\phi + Q\sin\phi &= R \\ P'\cos\phi + Q'\sin\phi &= R' \end{aligned}$$

is

$$R^2(P'^2+Q'^2) + R'^2(P^2+Q^2) - 2RR'(PP'+QQ') = (PQ'-QP')^2.$$

Whence, on substitution, we easily obtain

(calling $2a^2+2h^2+2c^2-b^2-d^2$, A, and $4hc+d^2-b^2$, B)

$$\begin{aligned} & r^6 \,.\, 4c^2 \\ & - r^4x \,.\, 8ach \\ & + r^4 \,.\, 4\{c(cA - hB) + a^2h^2\} \\ & - r^2x^2 \,.\, 16a^2c^2 \\ & - r^2x \,.\, 4a\{B(c^2-h^2) + 4a^2ch\} \\ & + x^3 \,.\, 32a^3ch \\ & + r^2 \,.\, \{4a^2h(hA - cB) + (cA - hB)^2 - 16a^2(c^2-h^2)^2\} \\ & + x^2 \,.\, 8a^2\{2a^2c^2 - c(cA - hB) - h(hA - cB) + 2(c^2-h^2)^2\} \\ & + x \,.\, 2a(hA - cB)(cA - hB - 4a^2c) \\ & + a^2(hA - cB)^2 = 0. \end{aligned}$$

It may be worth while to write this at full length: the result is

$$\begin{aligned} & r^6 \,.\, 4c^2 \\ & - r^4x \,.\, 8ach \\ & + r^4 \,.\, 4\{a^2(h^2+2c^2) + 2c^2(c^2-h^2) - b^2c(c-h) - d^2c(c+h)\} \\ & - r^2x^2 \,.\, 16a^2c^2 \\ & + r^2x \,.\, 4a\{(h^2-c^2)(4hc+d^2-b^2) - 4a^2ch\} \\ & + x^3 \,.\, 32a^3ch \end{aligned}$$

$$+r^2\,.\ \{4a^4(2h^2+c^2)-a^2[8(c^2-h^2)^2+4b^2(c-h)^2+4d^2(c+h)^2]$$
$$+[2c(c^2-h^2)-b^2(c-h)-d^2(c+h)]^2\}$$
$$+x^2\,.\ 8a^2\{b^2(c-h)^2+d^2(c+h)^2-2a^2h^2\}$$
$$-x\,.\ 2a\ \{2h(a^2+h^2-c^2)+b^2(c-h)-d^2(c+h)\}$$
$$\{2c(a^2+h^2-c^2)+b^2(c-h)+d^2(c+h)\}$$
$$+a^2\{2h(a^2+h^2-c)+b^2(c-h)-d^2(c+h)\}^2=0.$$

The same equation has been obtained by a slightly different method of elimination by Professor W. W. Johnson (*Messenger of Mathematics*, vol. V., 1875, p. 50).

The equation is so unmanageable in this form that some special assumption seems necessary in order to be able to trace the sextic curve; and the assumption in this paper is that $b=d$, $h=0$; in other words, the tracing point bisects the free link, while the fixed links are of equal length.

The curve now becomes

$$r^6+2r^4(a^2+c^2-b^2)-4r^2x^2a^2$$
$$+r^2\{a^2(a^2-2c^2-2b^2)+(c^2-b^2)^2\}$$
$$+4a^2b^2x^2=0.$$

This equation may be readily solved to give r in terms of θ, the vectorial angle, and the result (which may be easily obtained in other ways) is

$$r^2=a^2\cos 2\,\theta+b^2-c^2\pm 2a\sin\theta\,\sqrt{c^2-a^2\cos^2\theta},$$

or

$$\sin\theta=(a^2-c^2+b^2-r^2)/2a\,\sqrt{(b^2-r^2)}.$$

The minimum values of r^2 are when $\cos\theta=0, \sin\theta=1, \cos 2\,\theta=-1$, and in this case $r^2=b^2-(a+c)^2$. This expression being only positive if $b>(a+c)$ the division is at once suggested into the cases

I. $b>a+c$

II. $b=a+c$

III. $b<a+c$.

Before proceeding to the separate cases, we may notice that the maximum and minimum values of r^2 are given by the equation

$$rdr/d\theta=0=-a^2\sin 2\,\theta+a\cos\theta(c^2-a^2\cos 2\,\theta)/\sqrt{(c^2-a^2\cos^2\theta)},$$

whose only solution is $\cos\theta=0$; and those of y^2 by the equation

$$\sin\theta\cos\theta\{4a^2\cos^2\theta+b^2-c^2-3a^2$$
$$\pm a\sin\theta(3c^2+a^2-4a^2\cos^2\theta)/\sqrt{(c^2-a^2\cos^2\theta)}\}=0,$$

whose solutions are $\cos\theta=0, \sin\theta=0$, and

$$8a^4b^2\cos^4\theta+a^2b^2\{b^2-10c^2-ba^2\}\cos^2\theta+\{(a-c)^3+b^2a\}\{(a+c)^3-b^2c\}=0 \quad \ldots \quad (1)$$

or

$$16a^2\cos^2\theta=10c^2+6a^2-b^2\pm\{b^2+2(c^2-a^2)\}\sqrt{1+8(c^2-a^2)/b^2} \quad (2)$$

I. $b>a+c$.

We notice that if $a \ngtr c$, θ can have every value, otherwise the maximum value of $\cos\theta$ is c/a: we therefore divide this case into

(i.) $a<c$, (ii.) $a=c$, (iii.) $a>c$.

(i.) $a<c$.

Here θ may have every value giving always two values of r^2 except at $\theta=0$. We get two detached but intersecting ovals which must be separately described, as may be at once seen on drawing a diagram.

On referring to equation (1) we see that if $b^2>(a+c)^3/c$, the product of the values of $\cos^2\theta$ is negative, and the negative value of $\cos^2\theta$ is inadmissible, while from equation (2) it is clear that the positive value of $\cos^2\theta$ is greater than unity. Thus the only maximum and minimum heights are on the axis of y. The form of one branch of the curves is given in figure i. In this, and in all the other figures, the three numbers affixed denote the values of the constants a, b. c, taken in that order. As a rule the standard value of b is 12, but for clearness it is taken of various convenient magnitudes.

If $b^2<(a+c)^3/c$, both values of $\cos^2\theta$ are positive, and the smaller value is less than unity. This gives two maximum values of y on each loop, symmetrically situated with regard to the axis of y; while if $b^2=(a+c)^3/c$ these two maxima coalesce with the minimum between them, giving a flattish figure to the curve (see figures ii. and iii.), and a point where the tangent meets it in four coincident points.

(ii.) $c=a$.

In this case, since the two loops in general make (on opposite sides) an angle with the axis of x, where they cut, of

$$\tan^{-1}\{(a^2+b^2-c^2)/a\sqrt{(c^2-a^2)}\}$$

the two loops cut the axis of x at right angles, and therefore touch and may each be continued from the other, since the lines BC, AD when they lie along AB may be made to cross or not cross in continuing their motion, as may be desired.

The equation reduces to

$$r^2=b^2-2a^2\sin^2\theta\pm 2a^2\sin\theta\sqrt{(1-\cos^2\theta)},$$

which, by taking the top half of each with the bottom of the other, becomes

$$r^2=b^2$$
$$r^2=b^2-4a^2\sin^2\theta.$$

The latter is the inverse of an ellipse with regard to its centre, since $b > 2a$; and it becomes two circles if $b = 2a$.

This curve being written

$$y^2 = \{b^4 - (b^2 - 2r^2)^2\}/16a^2$$

the least value of r^2 is $b^2 - 4a^2, \theta = \pi/2$; and if this is $> b^2/2$, the greatest value of y^2 is when r^2 has this least value; this requires b^2 to be $> 8a^2$; but if $b^2 < 8a^2$, and therefore $b^2/2 > b^2 - 4a^2$, r^2 can be as small as $b^2/2$, and this then gives a *maximum* value of y—viz., $b^2/4a, \sin^2\theta = b^2/8a^2$; and a *minimum* value of y when $\theta = \pi/2$. Also when $b^2 = 8a^2$ these two coalesce giving a masked point of inflexion on the axis of y. See figures iv., v., vi.

It is interesting to notice that as a approaches c two disconnected equal portions of the loops become the circle, and the inverse of the conic; so that each loop degenerates into a semi-circle, and half of the inverse of the conic.

(iii.) $a > c$.

In this case $\cos\theta$ is limited, its maximum value being c/a.

If $b^2 > (a+c)^3/c$, we find that the values of $\cos^2\theta$ in equation (1) are one positive and less than c^2/a^2 (as may be at once verified by substituting c/a for $\cos\theta$ in the expression in the left which is then positive, while it is negative if $\cos\theta$ is put equal to zero), and the other negative. This gives us one minimum value of y^2, and its two positions are on the lower part of the upper loop and *vice versa*. See figure vii.

It is noteworthy that to pass from the case $a = c$, to the case $a > c$, we may regard the *upper* half of the circle as combined with the *upper* half of the inverse curve, whereas in passing to the case $a < c$, as seen above, the upper half of the circle is combined with the lower half of the inverse curve.

If $b^2 = (a+c)^3/c$ the equation for $\cos^2\theta$ becomes

$$8a^2\cos^4\theta + (b^2 - 10c^2 - 6a^2)\cos^2\theta = 0,$$

giving $$\cos^2\theta = 0 \text{ or } (a^2 + 3c^2)(3c - a)/8a^2c,$$

the last value is always less than c^2/a^2, and we thus have, in this case, an additional zero value of $\cos^2\theta$, and the tangent at the vertex meets the curve in four coincident points. If $3c > a$ there are two other symmetrical minima on each loop (fig. viii.), if $3c = a$ these also move up to the vertex where the tangent now meets the curve at six coincident points (fig. ix.), and if $3c > a$, these minima disappear (fig. x.).

The equation to the curve of fig. ix. is
$$x^6+3x^4(y^2-48c^2)+3x^2(y^2-48c^2)(y^2-36c^2)+y^2(y^2-48c^2)(y^2-60c^2)=0$$
from which it appears that the lines $y^2=48c^2$ each meet the curve in six coincident points, and the lower part of the curve approaches very closely to a straight line.

If $b^2<(a+c)^3/c$ clearly no other maxima or minima exist unless $b^2<10c^2+6a^2$ (fig. xi.); if $b^2<10c^2+6a^2$, we have to divide the cases into, from equation (2), (i.) $b^2>8(a^2-c^2)$ in which case both values if $\cos^2\theta$ are real and admissible giving four)symmetrical) maxima or minima heights on each loop (fig. xii.); (ii.) $b^2=8(a^2-c^2)$ in which case these coalesce two and two giving a point of inflexion with a horizontal tangent (fig. xiii.); (iii.) $b^2<8(a^2-c^2)$, in which case the points of inflexion are left but the horizontal tangent becomes oblique (fig. xiv.).

II. $b=a+c$.

This case naturally divides like the last into
$$\text{(i.) } a<c,\ \text{(ii.) } a=c,\ \text{(iii.) } a>c\,;$$
and in each case
$$r^2=2(a^2\cos^2\theta+ac\pm a\sin\theta\sqrt{c^2-a^2\cos^2\theta}),$$
or $\quad r=\sqrt{a(1+\cos\theta)(c+a\cos\theta)}\pm\sqrt{a(1-\cos\theta)(c-a\cos\theta)}$.

The equation to obtain the maximum height is now, besides $\cos\theta=0$, $16a^2\cos^2\theta=9c^2-2ac+5a^2\pm(3c-a)\sqrt{(c+a)(9c-7a)}$.

Case (i.) $a<c$.

Here the upper sign gives a value of $\cos^2\theta$ greater than unity, and the lower sign gives a positive root less than unity, as may be seen from the more general case I. (i.), or by comparing $(9c^2-2ac+5a^2)^2$, and $(9c^2-2ac-11a^2)^2$ with $(3c-a)^2(9c^2+2ac-7a^2)$.

Hence in this case there is one maximum height besides that on the axis (fig. xv.). The two minima heights on the axis here become zero, the two loops meet one another at the origin with a common vertical tangent there, so that there is a choice of path at the central point.

Case (ii.) $a=c$.

Here the pairs of tangents at the points of intersection on the axis of x, which were inclined to that axis, become coincident, and we simply get three circles (fig. xvi.).

Case (iii.) $a>c$,

In this case we have, as in I. (iii.), the condition $\cos^2\theta \not> c^2/a^2$; subject to this both values of r^2 are real. For the maximum height we have the same equation as in case (i.), namely, $16a^2\cos^2\theta = 9c^2 - 2ac + 5a^2 \pm (3c-a)\sqrt{(c+a)(9c-7a)}$, and the values of $\cos^2\theta$ are real if $9c$ is equal to or greater than $7a$, or if $3c=a$. Now if $7a<9c$, it is easily seen that both values of $\cos^2\theta$ are admissible, and we have (fig. xvii.) four (symmetrical) maxima and minima: if $7a=9c$, these coincide pair and pair, giving a point of inflexion with a horizontal tangent (fig. xviii.).

If $7a>9c$ (fig. xix.) these points disappear, while if $a=3c$, the values of $\cos^2\theta$ though real give imaginary values of r^2, and the figure resembles that just referred to.

III. $b<a+c$.

Writing our original equations in the form

$$r^2 = a^2+b^2-c^2-2a\sin\theta(a\sin\theta \pm \sqrt{c^2-a^2\cos^2\theta}),$$
$$\sin\theta = \{c^2-(a^2+b^2)+r^2\}/2a\sqrt{b^2-r^2},$$

we shall find a sub-division $c^2 > = < a^2+b^2$ convenient; but we shall keep to the other division of $a > = < c$ as before.

Case (i.) $a<c$.

If $a^2+b^2<c^2$, $\sin\theta$ increases with r, and lies between $\{c^2-a^2-b^2\}/2ab$ and 1, and each value of $\sin\theta$ gives only one positive value of r^2: the least value of r^2 is zero, $\sin\theta = \{c^2-a^2-b^2\}/2ab$, and the greatest is $b^2-(a-c)^2$, $\sin\theta=1$: the curve consists of a simple figure of 8 (fig. xx.).

If $a^2+b^2=c^2$, the two loops touch with the axis of x for common tangent meeting the curve in six coincident points as in fig. xxi.: the equation in fact reduces to $4a^2y^2(b^2-r^2)=r^6=(x^2+y^2)^6$.

If $a^2+b^2>c^2$, the values of r^2 are sometimes both positive, sometimes not: the limiting value of θ is clearly given by

$$a^2+b^2-c^2 \not< 2a\sin\theta(a\sin\theta + \sqrt{c^2-a^2\cos^2\theta}),$$

or, in other words, $\sin\theta$ must not be greater than the value of θ that makes r^2 vanish: thus one available value of r^2 exists for all values of θ and two values form $\sin\theta=0$ to $\sin\theta=(a^2+b^2-c^2)/2ab$.

The maximum height is found as before, and refers to the loop (fig. xxi.). In this case the curve is described by a single operation without any choice of direction at any point, or break of continuity.

Case (ii.) $a=c$.

Here we get at once the equations

$$r^2=b^2, \text{ and } r^2=b^2-4a^2\sin^2\theta,$$

which last, since $b<2a$, is the inverse of a hyperbola that becomes rectangular when $b^2=2a^2$ (fig. xxiii.).

Case (iii.) $a>c$.

In this case $\cos\theta<c/a, \sin\theta>\sqrt{1-c^2/a^2}$, and since we have

$$\sin\theta=\{(a^2-c^2)+(b^2-r^2)\}/2a\sqrt{b^2-r^2},$$

we see that $\sin\theta$ decreases with b^2-r^2 until, if possible, $b^2-r^2=a^2-c^2$, at which point it begins to increase again until $\sin\theta=1$.

When $r=0, \sin\theta=(a^2+b^2-c^2)/2ab$, which is always an admissible value: and between $\sin\theta=(a^2+b^2-c^2)/2ab$ and $\sin\theta=\sqrt{1-c^2/a^2}$ there are always two values of r^2, it only remains to see whether r^2 can ever become $b^2+c^2-a^2$: this of course depends on whether $a^2<=>b^2+c^2$.

If $a^2<(b^2+c^2)$ there are two real values of r between these limits, and one beyond, *i.e.*, from $\sin\theta=(a^2+b^2-c^2)/2ab$ to $\sin\theta=1$. On referring to the equations (1) and (2) we find that if $b^2>8(a^2-c^2)$ (and therefore $b^2+c^2>a^2$) there are four symmetrical maximum and minimum heights (fig. xxiv.); if $b^2=8(a^2-c^2)$ there are only two, through the coincidence of two pairs forming a point of inflexion with a horizontal tangent (fig. xxv.), while if $b^2<8(a^2-c^2)$ there are none (fig. xxvi.).

If $a^2=b^2+c^2$, the two limits coincide and we have a curve with a very approximately straight portion near the node: this, in fact, (fig. xxvii.) is Watts' parallel motion.

If $a^2>b^2+c^2$, the value of θ decreases till $r^2=0$, but cannot decrease further (fig. xxviii.) through the disappearance of the limit $\sin\theta=\sqrt{1-c^2/a^2}$.

In all the descriptions of these curves the upper half only has been referred to; and from the obvious symmetry it has only been necessary to discuss values of θ less than $\pi/2$. It is clear that we might from the beginning have proceeded under the heads $a<c$, $a=c$, $a>c$; but some consideration has led to the adoption of the process of this paper. There being two independent ratios, there are numerous cross connections between the curves, but these are best seen by the help of the diagrams.

INDEX TO DIAGRAMS.

	a	b	c	*No.*
I. $b>a+c$.				
(i.) $a<c$.				
All values of θ are here admissible with either sign; r never vanishes, and the curve has two separate loops cutting on the axis of x.				
$b^2>(a+c)^3/c$, a maximum height at $\theta=\pi/2$ only	3	12	6	i.
$b^2=(a+c)^3/c$, three coincident maxima there	$\frac{63}{16}$	12	$\frac{81}{16}$	ii.
$b^2<(a+c)^3/c$, two maxima on each loop, besides those at $\theta=\pi/2$, ...	5	12	6	iii.
(ii.) $a=c$.				
The loops now touch at $\theta=0$, and one-half of each loop becomes a semi-circle; the other, half the inverse of a conic.				
$b^2>8a^2$, a special case of i., ...	3	12	3	iv
$b^2=8a^2$, do. ii., ...	$3\sqrt{2}$	12	$3\sqrt{2}$	v.
$b^2>8a^2$, do. iii., ...	$\frac{21}{4}$	12	$\frac{21}{4}$	vi.
(iii.) $a>c$.				
Here $a\cos\theta \not> c$, and the curve is bounded by two radii; we have two detailed loops.				
$b^2<(a+c)^3/c$, the upper half of the circle in iv. joins the upper half of the inverse, and *vice versa*, ...	8	24	4·38	vii.
$b^2=(a+c)^3/c$, the tangent at the lower vertex meets the curve at four points.				

	a	b	c	*No.*
$3c>a$, two other minima on each loop	6·84	18	4.5	viii.
$3c=a$, these move to the vertex, where a tangent meets the curve in six points, ...	$13\frac{1}{2}$	36	$4\frac{1}{2}$	ix.
$3c>a$, these minima disappear, ...	25·44	72	6	x.
$b^2<(a+c)^3/c$, we must consider in three cases, $b^2 \nless 10c^2+6a^2$, we have no other maxima, ...	25·5	72	6	xi.
$b^2<10c^2+6a^2, >8(a^2-c^2)$, there are four other maxima on each loop, ...	5	12	4	xii.
$b^2<10c^2+6a^2, =8(a^2-c^2)$, they coalesce giving two points of inflexion with a horizontal double tangent, ...	$8\frac{1}{4}$	18	$5\frac{1}{4}$	xiii.
$b^2<10c^2+6a^2, <8(a^2-c^2$, the points of inflexion are left, but the horizontal tangent goes, ...	9	18	6	xiv.
II. $b=a+c$.				
Here one branch of the curve has a zero value of r, all values of θ are possible if $a \ngtr c$.				
i. $a<c$.				
The whole curve may be continuously described, ...	$4\frac{1}{2}$	12	$7\frac{1}{2}$	xv.
ii. $a=c$.				
Three circles, ...	6	12	6	xvi
iii. $a>c$.				
In this case, $\cos\theta \ngtr c/a$,				
$7a<9c$, gives four other maxima, compare xii., ...	$6\frac{1}{4}$	12	$5\frac{3}{4}$	xvii.
$7a=9c$, these reduce to two as in xiii., ...	$6\frac{3}{4}$	12	$5\frac{1}{4}$	xviii.
$7a>9c$, and here disappear as in xiv., ...	8	12	4	xix.

	a	b	c	*No.*
III. $b<a+c$.				
Here there is a zero value of r, but not every value of θ gives two admissible values of r^2.				
i. $a<c$.				
One value of r for each value of θ and *vice versa*.				
If $a^2+b^2<c^2$, we have a looped curve, and $\sin\theta$ must not be less than $\{c^2-(a^2+b^2)\}/2ab$, ...	6	12	15	xx.
If $a^2+b^2=c^2$, this angle becomes zero, and the loops touch instead of cutting, ...	6	12	$6\sqrt{5}$	xxi.
If $a^2+b^2>c^2$, some values of θ have two values of r; and two other nodes on the axis of x are developed, ...	6	12	9	xxii.
ii. $a=c$.				
Since $a^2+b^2>c^2$, this resembles xxii., except that the loops touch at the outer nodes, giving a double genesis, ...	$6\sqrt{2}$	12	$6\sqrt{2}$	xxiii.
iii. $a>c$.				
The curve is limited by the condition as regards $\cos\theta$; and we have if $a^2<b^2+c^2$ one value of r from $\sin\theta=(a^2+b^2-c^2)/2ab$ to $\theta=\pi/2$, and two from the latter limit to $\cos\theta=c/a$.				
$b^2>8(a^2-c^2)$, four maximum heights besides those on the axis, compare xii. and xvii., ...	7	12	6	xxiv.
$b^2=8(a^2-c^2)$, a horizontal inflexional tangent, compare xiii. and xviii., ...	$9\frac{1}{2}$	12	$8\frac{1}{2}$	xxv.
$b^2<8(a^2-c^2)$, the maxima vanish, compare xiv. and xix., ...	9	12	6	xxvi.
$a^2=b^2+c^2$ gives us Watt's motion, and the limits in the general case coincide, ...	15	9	12	xxvii.
$a^2>(b^2+c^2)$, the lesser limit disappears, and we have one value of r for each value of θ and *vice versa*, ...	20	12	12	xxviii.

On some theorems in the theory of numbers.

By R. E. ALLARDICE, M.A.

The number of groups of n which may be selected from $2n$ is $2n(2n-1)\ldots(n+1)/n!$ But make the $2n$ into two groups of n, and select r out of the first and $n-r$ out of the second. This gives $[n(n-1)\ldots(n-r+1)/r!]+[n(n-1)\ldots(r+1)/(n-r)!]$ ways of thus making a group of n. Hence

$$* \; 2n(2n-1)\ldots(n+1)/n! = 1+n^2+[n(n-1)/2!]^2+ \; \ldots \quad \ldots \qquad (1).$$

$$\therefore\; 2n(2n-1)\ldots(n+1)/n!-2$$
$$= n^2\{1^2+[(n-1)/2!]^2+[(n-1)(n-2)/3!]^2+\ldots\}$$
$$= n^2\{P_1^2+P_2^2+\ldots+P_{n-1}^2\} \text{ (say).}$$

We shall now show that $P_1^2+P_2^2+ - - +P_{n-1}^2$ is divisible by n, if n be prime.

$P_r \equiv P_s$ (mod. n.), if $r+s=n$, but not otherwise. For if $P_r \equiv P_s$ ($r>s$), then

$$(n-1)(n-2)\ldots(n-r+1)/r! - (n-1)(n-2)\ldots(n-s+1)/s! \equiv 0\,;$$
$$\therefore \frac{(n-1)(n-2)\ldots(n-s+1)}{s!}\left\{\frac{(n-s)(n-s-1)\ldots(n-r+1)}{(s+1)(s+2)\ldots \quad r}-1\right\}\equiv 0\,;$$
$$\therefore\; (n-s)(n-\overline{s+1})(n-\overline{s+2})\ldots(n-\overline{r-1})-(s+1)(s+2)\ldots r\equiv 0\,;$$
$$\therefore\quad \pm s(s+1)(s+2)\ldots(r-1)-(s+1)(s+2)\ldots r\equiv 0\,;$$
$$\therefore\quad -(s+1)(s+2)\ldots(r-1)(\mp s+r)\equiv 0\,;$$

and this is true if $r+s=n$ (otherwise obvious) and not in any other case. [If $r+s=n$, then $r-s=n-2s$, which is odd, and the lower sign is to be taken where the double sign is printed.]

It is obvious that P_r+P_s is not divisible by n; and hence if we divide $P_1, P_2, \ldots P_{(n-1)/2}$ by n, we must get for remainders either 1 or $n-1$ and either 2 or $n-2$ and so on.

Now since $(n-r)^2=n^2-2nr+r^2\equiv r^2$ (mod. n), we must have

$$P_1^2+P_2^2+\ldots P_{(n-1)/2}^2 \equiv 1^2+3^2+5^2\ldots+(n-2)^2$$
$$= n(n-1)(n-2)/6\,;$$

which is divisible by n if n be any prime except 2 or 3.

From this, and the identity (1), it follows that

$$(2n-1)(2n-2)\ldots(n+1)-(n-1)(n-2)\ldots 1\equiv 0 \text{ (mod. } n^3).$$

* The use of this identity was suggested to me by Professor Tait.

We shall next show that $\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)(n-1)!$ is divisible by n^2.*

We have

$$\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)(n-1)!=\left(\frac{n}{1.(n-1)}+\frac{n}{2.(n-2)}+\ldots\right)(n-1)!$$

Hence we have to show that $\left(\frac{1}{1.(n-1)}+\frac{1}{2.(n-2)}+\ldots\right)(n-1)!$ is exactly divisible by n.

Assume $(n-1)!/1.(n-1)=a_1$, $(n-1)!/2.(n-2)=a_2$, &c.

Then

$$\begin{aligned}(r+1)^2a_{r+1}-r^2a_r &= (r+1)^2(\overline{n-1}!)/(r+1)(n-r-1)-r^2(\overline{n-1}!)/r(n-r)\\ &= \{(r+1)/(n-r-1)-r/(n-r)\}(n-1)!\\ &= (n!)/(n-r)(n-r-1)\equiv 0 \pmod{n}.\\ \therefore\ (r+1)^2a_{r+1} &\equiv r^2a_r\\ &\equiv (r-1)^2a_{r-1}\\ &\ldots\quad\ldots\quad\ldots\\ &\equiv 1^2.a_1\\ &\equiv 1 \qquad \text{(by Wilson's theorem).}\end{aligned}$$

Hence we may write

$$\begin{aligned}a_1 &= n\mu_1+1\\ 2^2a_2 &= n\mu_2+1\\ \ldots&\quad\ldots\quad\ldots\\ m^2a_m &= n\mu_m+1 \qquad \text{(where } m=(n-1)/2)\end{aligned}$$

$\therefore\ (m!)^2\Sigma a_r=\mathrm{P}.n+(m!)^2(1/1^2+1/2^2+\ldots+1/m^2)$.

Now, if we assume $(m!)/r\equiv\alpha_r$, we may easily show that $\alpha_r\pm\alpha_s$ is not divisible by n, and hence that

$$\alpha_1^2+\alpha_2^2+\ldots+\alpha_m^2\equiv 0 \pmod{n},$$

which proves the theorem.

Consider again the result

$$\mathrm{A}=(2n-1)(2n-2)\ldots(n+1)-(n-1)(n-2)\ldots1\equiv 0 \pmod{n^3}.$$

This gives

$$\begin{aligned}\mathrm{A} &= (n+\overline{n-1})(n+\overline{n-2})\ldots(n+1)-(n-1)(n-2)\ldots1\\ &= n^{n-1}+p_1n^{n-2}+\ldots+p_{n+3}n^2+p_{n-2}n,\end{aligned}$$

where
$$\begin{aligned}p_{n-3} &= (\overline{n-1}!)\Sigma(1/rs)(r+s)\\ p_{n-2} &= (\overline{n-1}!)\Sigma(1/r).\end{aligned}$$

---

* Compare a paper by Mr Leudesdorf, in the *Proceedings of the Lond. Math. Soc.* for 1889, p. 199—a paper which I did not see till after the above was written.

Now p_{n-2} is divisible by n^2, and hence p_{n-3} is divisible by n.

This theorem may also be proved in the following manner:—

We have $2(\overline{n-1}!)\Sigma(1/rs)$

$$=\{(\overline{n-1}!)/1.2+(\overline{n-1}!)/1.3+\ldots+(\overline{n-1}!)/1.(n-1)\}(=\text{P}_1)$$
$$+\{(\overline{n-1}!)/2.1+(\overline{n-1}!)/2.3+\ldots+(\overline{n-1}!)/2.(n-1)\}(=\text{P}_2)$$
$$+\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

Now consider the terms of P_r, namely,

$$(\overline{n-1}!)/r.1,(\overline{n-1}!)/r.2,\ldots\ldots(\overline{n-1}!)/r.(r-1),(\overline{n-1}!)/r.(r+1),\ \&c.$$

No two of these can be congruent; and

$$(\overline{n-1}!)/r.p+(\overline{n-1}!)r.(n-p)=n!/r.p.(n-p)\equiv 0\ (\text{mod. } n).$$

Hence if we divide each of the terms of P_r by n, we get as remainders all the numbers 1, 2, $3\ldots n-1$, with the exception of that number which is complimentary to a_r where a_r

$$\equiv(\overline{n-1}!)/r.(n-r)\ (\text{mod. } n).$$

Hence the sum of all the remainders in $2\Sigma(n-1!)/rs$

$$=(n-1)(1+2+\ldots+\overline{n-1})-2(a_1+a_2+\ldots+a_{(n-1)})$$
$$=(n-1)^2n/2-2\{1/1.(n-1)+1/2.(n-2)+\ldots\}$$

which is divisible by n.

The theorem that the sum of the reciprocals of the numbers $1, 2,\ldots\overline{n-1}$, is divisible by n^2, when n is a prime, may be extended to the sum of the m^{th} powers of these numbers, where m is an integer, positive or negative.

Let $\text{S}_m=1^m+2^m+\ldots(n-1)^m$; it being understood that if m is negative $(=-l)$, the sum of the powers is to be multiplied by $(\overline{n-1}!)^l$, so that it may be made integral.

Since, when n is prime the equation

$$(x-1)(x-2)\ldots(x-\overline{n-1})-x^{n-1}+1\equiv 0\ (\text{mod. } n)$$

has $(n-1)$ incongruent solutions, each co-efficient is divisible by n. Hence, if m is positive, S_m is divisible by n, unless m is a multiple of $(n-1)$.

Suppose now that m is an odd positive integer and $n\neq 2$; then

$$2\ \text{S}_m=2\Sigma a^m=\Sigma(a^m+\overline{n-a}^m)$$
$$=\Sigma\{a^m+n^m-{}_m\text{C}_1n^{m-1}a+\ldots+{}_m\text{C}_1na^{m-1}-a^m\}$$
$$\equiv n\Sigma_m\text{C}_1a^{m-1}\equiv nm\text{S}_{m-1};$$

and $\quad \text{S}_{m-1}\equiv 0$ (mod. n), unless $m-1$ is a multiple of $n-1$;

$\therefore \quad \text{S}_m\equiv 0$ (mod. n^2), unless $m-1$ is a multiple of $n-1$;

and the theorem is true even in this last case if m is a multiple of n.

Now consider

$$S_{-m} = \{1 + 1/2^m + 1/3^m + \ldots + 1/(n-1)^m\}(\overline{n-1}!)^m.$$

We have

$$\begin{aligned} 2\,S_{-m} &= (\overline{n-1}!)^m \Sigma\{1/r^m + 1/(n-r)^m\} \\ &= (\overline{n-1}!)^m \Sigma \frac{(n-r)^m + r^m}{r^m(n-r)^m} \\ &= (\overline{n-1}!)^m \Sigma \frac{n^m - {}_mC_1 n^{m-1} r \ldots + {}_mC_1 n r^{m-1}}{r^m(n-r)^m} \\ &= (\overline{n-1}!)^m \Sigma \frac{Pn^2 + {}_mC_1 n}{r(n-r)^m}. \end{aligned}$$

We have thus to show that ${}_mC_1(\overline{n-1}!)^m\Sigma\{1/r(n-r)^m\}$ is divisible by n. We shall suppose, for the sake of clearness, that m is less than n; but the following method will be applicable, even if m be greater than n.

Assume $\quad (\overline{n-1}!)^m/\{(n-r)r^m\} \equiv a_r \pmod{n}$

$\therefore \quad (\overline{n-1}!)^m \equiv a_r(n-r)r^m$

$\therefore \quad a_r(n-r)r^m \equiv -1 \quad$ (by Wilson's theorem).

Now since $\quad (n-r)^{n-1} \equiv r^{n-1} \equiv 1$, we get

$\quad a_r \equiv -r^{n-m-1}(n-r)^{n-2}$

$\therefore \quad a_r \equiv r^{n-m-1} r^{n-2} \pmod{n}$

$\quad \equiv r^{n-m-2}$

Hence $\quad \Sigma a_r \equiv \Sigma r^{n-m-2}$

$\quad \equiv 0$, if $n - m - 2 \neq 0$.

It follows that S_{-m} is divisible by n_2, m being subject to the restriction $n - m - 2$ be not zero. If we remove the condition that m is to be less than n, we shall easily find that the general restriction as to the value of m, is that $m + 1$ must not be a multiple of $n - 1$.

In the paper referred to before in a footnote, Mr Leudesdorf considers the case where n is not prime and S_m denotes the sum of the m^{th} powers of the numbers less than n and prime to it. His method however cannot be considered rigorous, as it involves the use of divergent series.

Note on normals to conics.

By R. H. Pinkerton, M.A.

1. The following condition may be new; it does not appear in any of the books:—

The condition that the straight line

$$lx + my + n = 0 \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (1)$$

should be a *normal* to the general conic,

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots \quad \dots \quad \dots \quad \dots \qquad (2),$$

referred to rectangular axes, is

$$\Sigma(al^2 + 2hlm + bm^2) = \Delta(l^2 + m^2)^2 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (\alpha),$$

where Σ is written for

$$\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2 + 2\mathrm{F}mn + 2\mathrm{G}nl + 2\mathrm{H}lm,$$

and Δ, A, B, C, F, G, H have their usual meanings.

We know* that the equation to the tangents to S at the points where (1) cuts S, is

$$S\Sigma = \Delta(lx + my + n)^2,$$

or

$$x^2(a\Sigma - \Delta l^2) + 2xy(h\Sigma - \Delta lm) + y^2(b\Sigma - \Delta m^2)$$
$$+ \text{terms of lower degree} = 0 \quad \dots \quad \dots \quad \dots \qquad (3).$$

Now the straight line (1) will be normal to S if it is perpendicular to one of the straight lines (3). The condition for this is

$$l^2(a\Sigma - \Delta l^2) + 2lm(h\Sigma - \Delta lm) + m^2(b\Sigma - \Delta m^2) = 0,$$

or

$$\Sigma(al^2 + 2hlm + bm^2) = \Delta(l^4 + 2l^2m^2 + m^4), \text{ which is } (\alpha).$$

2. If the line (1) be given as passing through a given point (α, β) its equation will be

$$lx + my - (l\alpha + m\beta) = 0.$$

If in the condition (α) we substitute $-l\alpha - m\beta$ for n, we shall get the following biquadratic in l/m—

$$(al^2 + 2hlm + bm^2) \times$$
$$[\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}(l\alpha + m\beta)^2 - 2\mathrm{F}m(l\alpha + m\beta) - 2\mathrm{G}l(l\alpha + m\beta) + 2\mathrm{H}lm]$$
$$= \Delta(l^2 + m^2)^2,$$

or

$$l^4[a(\mathrm{A} + \mathrm{C}\alpha^2 - 2\mathrm{G}\alpha) - \Delta]$$
$$+ 2l^3m[a(\mathrm{C}\alpha\beta - \mathrm{F}\alpha - \mathrm{G}\beta + \mathrm{H}) + h(\mathrm{A} + \mathrm{C}\alpha^2 - 2\mathrm{G}\alpha)]$$
$$+ l^2m^2[a(\mathrm{B} + \mathrm{C}\beta^2 - 2\mathrm{F}\beta) + b(\mathrm{A} + \mathrm{C}\alpha^2 - 2\mathrm{G}\alpha)$$
$$+ 4h(\mathrm{C}\alpha\beta - \mathrm{F}\alpha - \mathrm{G}\beta + \mathrm{H}) - 2\Delta]$$
$$+ 2lm^3[b(\mathrm{C}\alpha\beta - \mathrm{F}\alpha - \mathrm{G}\beta + \mathrm{H}) + h(\mathrm{B} + \mathrm{C}\beta^2 - 2\mathrm{F}\beta)]$$
$$+ m^4[b(\mathrm{B} + \mathrm{C}\beta^2 - 2\mathrm{F}\beta) - \Delta] = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (\beta).$$

This biquadratic gives the directions of the four normals which can be drawn from the point (α, β) to the conic S.

3. If (x, y) be any point on any one of the four normals from (α, β) to the conic S, then

$$l(x - \alpha) + m(y - \beta) = 0$$

---

* *Salmon's Conics*—Fifth Edition—Art. 374.

if l/m be one of the roots of (β).

Hence $$l/(y-\beta) = -m/(x-a).$$

If then for l and m we substitute in (β) $y-\beta$ and $-(x-a)$ respectively, we shall obtain the equation to the four normals which can be drawn from (a,β) to the conic S.

4. It may be worth considering what the condition (a) becomes for the circle—

$$x^2+y^2+2gx+2fy+c=0.$$

Here $$a=b=1\ ;\ h=0$$

$$\Delta=c-f^2-g^2,$$

$$A=c-f^2, B=c-g^2, C=1,$$

$$F=-f,\quad G=-g,\quad H=fg.$$

Hence in this case (a) becomes

$$(l^2+m^2)[l^2(c-f^2)+m^2(c-g^2)+n^2-2fmn-2gnl+2fglm]$$
$$=(l^2+m^2)^2(c-f^2-g^2).$$

Dividing by l^2+m^2, we obtain

$$c(l^2+m^2)-f^2l^2-g^2m^2+n^2-2fmn-2gnl+2fglm$$
$$=c(l^2+m^2)-f^2l^2-f^2m^2-g^2l^2-g^2m^2,$$

or $$g^2l^2+f^2m^2+2fglm-2n(gl+fm)+n^2=0,$$

or $$(gl+fm-n)^2=0.$$

Hence the condition that $lx+my+n=0$ should be a normal to the circle is $$gl+fm-n=0.$$

But this is the tangential equation to the point $(-g, -f)$, the centre of the circle.

Third Meeting, 10th January 1890.

George A. Gibson, Esq., M.A., Ex-President, in the Chair.

Note on a curious operational Theorem.

By Professor Tait.

The idea in the following note is evidently capable of very wide development, but it can be made clear by a very simple example.

Whatever be the vectors a, β, γ, δ, we have always

$$V.Va\beta V\gamma\delta = aS.\beta\gamma\delta - \beta S.a\gamma\delta.$$

But vector operators are to be treated in all respects like vectors, provided each be always kept *before* its subject.

Let $$\sigma = i\xi + j\eta + k\zeta,$$
where ξ, η, ζ are functions of x, y, z; and let
$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$
as usual. Also let σ_1, ∇_1 be their values when x_1, y_1, z_1 are put for x, y, z.

Then by the first equation, attending to the rule for the place of an operator,
$$\text{V}.\text{V}\nabla\sigma\text{V}\nabla_1\sigma_1 = \nabla\text{S}.\sigma\nabla_1\sigma_1 - \text{S}(\nabla_1\sigma_1\nabla)\sigma.$$

If we suppose the operations to be completed, and *then* make $x_1 = x$, $y_1 = y$, $z_1 = z$, the left-hand member must obviously vanish. So therefore must the right.

That is :— $$\nabla\text{S}.\sigma\nabla_1\sigma_1 = \text{S}(\nabla_1\sigma_1\nabla)\sigma\,;$$
if when the operations are complete, we put $\sigma_1 = \sigma, \nabla_1 = \nabla$.

In Cartesian co-ordinates this is equivalent to three equations, of the same type. I write only one, viz. :—
$$\frac{d}{dx}\begin{vmatrix} \xi & \eta & \zeta \\ \frac{d}{dx_1} & \frac{d}{dy_1} & \frac{d}{dz_1} \\ \xi_1 & \eta_1 & \zeta_1 \end{vmatrix} = \begin{vmatrix} \frac{d}{dx_1} & \frac{d}{dy_1} & \frac{d}{dz_1} \\ \xi_1 & \eta_1 & \zeta_1 \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \end{vmatrix}\xi,$$
if, *after* operating, we put $x_1 = x$, $\xi_1 = \xi$, &c., &c.

On a property of odd and even polygons.

By R. E. Allardice, M.A.

The property referred to comes to light on consideration of the problem, "To inscribe in a given n-gon the n-gon of minimum perimeter."

Triangle.

Let us consider first the case of the triangle (fig. 29). If ABC is the given triangle and DEF the inscribed triangle of minimum perimeter, it is obvious that we must have $\angle$ FDB = $\angle$ EDC($=\alpha$, say), $\angle$ DEC = $\angle$ FEA($=\beta$), $\angle$ EFA = $\angle$ DFB($=\gamma$). This condition is satisfied if D,E,F, are the feet of the perpendiculars from the oppo-

site vertices; and it is usually assumed that the triangle so obtained, the pedal triangle, must therefore be the triangle required; in other words, the assumption is made that only one inscribed triangle may be constructed, whose sides shall be equally inclined to the sides of ABC. This assumption is as a matter of fact correct in the case of the triangle; but it can hardly be considered not to require proof, as the corresponding assumption would be wrong in the case of the quadrilateral.

We might proceed to determine DEF as follows :—

We have
$$\alpha+\beta+C=\pi,$$
$$\beta+\gamma+A=\pi,$$
$$\gamma+\alpha+B=\pi;$$
from which we get
$$\alpha=A,\ \beta=B,\ \gamma=C.$$

To calculate CD, put $CD=x$, $AE=y$, $BF=z$, then
$$x\sin\alpha=(b-y)\sin\beta,$$
$$y\sin\beta=(c-z)\sin\gamma,$$
$$z\sin\gamma=(a-x)\sin\alpha;$$
whence
$$x\sin\alpha=b\sin\beta-c\sin\gamma+(a-x)\sin\alpha;$$
from which we get $x=(a^2+b^2-c^2)/2a$; which shows that D is the foot of the perpendicular from A on BC.

QUADRILATERAL.

Consider next the case of the quadrilateral (fig. 30),

Let $\angle APS=\angle BPQ=\alpha$; $\angle BQP=\angle CQR=\beta$; etc.

If now we try to determine α as we did in the case of the triangle, we get the equations
$$\alpha+\beta+B=\pi,\qquad \beta+\gamma+C=\pi,$$
$$\gamma+\delta+D=\pi,\qquad \delta+\alpha+A=\pi,$$
whence
$$B-C+D-A=0;$$
and the problem is therefore impossible unless the given quadrilateral be cyclic; that is to say, it is in general impossible to inscribe in a given quadrilateral a quadrilateral each pair of consecutive sides of which shall be equally inclined to the side of the given quadrilateral in which they meet.

If the given quadrilateral be cyclic there are an infinite number of inscribed quadrilaterals satisfying the given condition.

The investigation that determined the position of the vertices in the case of the triangle determines the inclination of the sides in the case of the quadrilateral.

Using a corresponding notation in this case, we have as before

$$x\sin\alpha = (b-y)\sin\beta, \qquad z\sin\gamma = (d-w)\sin\delta,$$
$$y\sin\beta = (c-z)\sin\gamma, \qquad w\sin\delta = (a-x)\sin\delta;$$

whence $$a\sin\alpha - b\sin\beta + c\sin\gamma - d\sin\delta = 0.$$

Let now α' denote the angle subtended at the circumference of the circumcircle by the side a, β' the angle subtended by the side b; and so on. Then $a = 2R\sin\alpha'$, and therefore

$$\sin\alpha'\sin\alpha - \sin\delta'\sin\delta = \sin\beta'\sin\beta - \sin\gamma'\sin\gamma.$$

But $$\sin\alpha' = \sin(A+\delta'); \ \sin\beta' = \sin(C+\gamma'),$$
$$\sin\delta = \sin(A+\alpha), \quad \sin\gamma = \sin(C+\beta);$$

$$\therefore \sin(A+\delta')\sin\alpha - \sin\delta'\sin(A+\alpha) = \sin(C+\gamma')\sin\beta - \sin\gamma'\sin(C+\beta);$$
$$\therefore \quad \sin A\sin(\alpha-\delta') = \sin C\sin(\beta-\gamma');$$
$$\text{and } \sin A = \sin C, \quad \therefore \sin(\alpha-\delta') = \sin(\beta-\gamma').$$

Hence either $\alpha - \delta' = \pi - (\beta - \gamma')$, and therefore $\alpha + \beta = \pi + \gamma' + \delta'$, which is impossible; or $\alpha - \delta' = \beta - \gamma'$ and $\therefore \alpha - \beta = \delta' - \gamma'$;
and we have also $$\alpha + \beta = \pi - \gamma' - \delta',$$
$$\therefore \alpha = \pi/2 - \gamma', \ \beta = \pi/2 - \delta'.$$

A geometrical investigation may also be given, in the following manner.

Let ABCD (fig. 31) be the given quadrilateral. Take U any point in DA; K the image of U in AB; L the image of K in BC; M the image of L in CD. Join MU, XL, WK, VU. Then it may easily be shown, that if ABCD is cyclic, UVWX has its adjacent side equally inclined to the sides of ABCD.

If U move along DA, the locus of K will be AR, the locus of L will be RS, the locus of M will be SM.

Now $\angle$ SMU $= \angle$ SLX $= \angle$ VKA $= \angle$ VUA $= \angle$ XUD; $\therefore$ SM is parallel to DA.

Now let U′ be another point in DA; K′, L′, M′, the points corresponding to K, L, M. Then UU′ = KK′ = LL′ = MM′. Hence MM′U′U is a parallelogram; the direction of UM is invariable; and UM which is equal to the perimeter of the inscribed quadrilateral is of constant length. [It should be noted that if the quadrilateral UVWX be crossed, one of its sides has to be considered negative.]

Having shown that the directions of the sides of the inscribed quadrilateral are invariable, and that the primeter is constant, we may make use of a particular case to determine the directions of the sides and the length of the perimeter.

Let one of the vertices of the inscribed quadrilateral (fig. 32) coincide with D, so that PQ is equal to the primeter.

The triangles KCD and HAD are similar, and therefore the triangles KDH and CDA are similar; hence

$$\mathrm{KH:CA} = \mathrm{KD:DC} = \sin \mathrm{C}$$
$$\therefore\ \mathrm{KH} = \mathrm{CA}\sin\mathrm{C}$$
$$\therefore\ \mathrm{PQ} = 2\mathrm{KH} = 2\mathrm{CA}\sin\mathrm{C} = hk/\mathrm{R},$$

where h and k are the diagonals of the quadrilateral and R is the radius of the circumcircle.

Denoting DMH by a, as before, we get

$$\pi/2 - a = \angle \mathrm{Q} = \angle \mathrm{KHD} = \angle \mathrm{CAD};$$
$$\therefore\ a = \pi/2 - \mathrm{CAD};$$

that is, a is the complement of the angle subtended at the circumference by the side of ABCD opposite to that in which the vertex of a lies.

It might be thought that there must in every case be some minimum inscribed quadrilateral. But if the given quadrilateral is not cyclic, the minimum inscribed quadrilateral is one which has negative infinity for the length of its perimeter. If the inscribed quadrilateral be restricted to be non-crossed, the minimum one will not be determinable by a general method.

It will be seen how it is that some sides may have to be considered negative, by consideration of figs. 33 and 34, where A′ and B′ are the geometrical images of A and B. In fig. 33, AP + PQ + QB (= A′B′) is a minimum; while if we make the same construction in fig. 34 we only get AP + PQ + QB equal to A′B′ and a minimum if we consider PQ negative.

We may also determine the value of the perimeter without considering merely a limiting case.

In fig. 30, let $\mathrm{BP} = x$, $\mathrm{BQ} = y$, $\mathrm{DR} = z$, $\mathrm{DS} = w$; $\mathrm{PQ} = k$, $\mathrm{QR} = l$, $\mathrm{RS} = m$, $\mathrm{SP} = n$; $\angle \mathrm{BPQ} = a$, etc.

Then $\quad k = x\cos a + y\cos\beta, \qquad l = (b - y)\cos\beta + (c - z)\cos\gamma,$
$\qquad m = z\cos\gamma + w\cos\delta, \qquad n = (d - w)\cos\delta + (a - x)\cos a.$

Hence, if p denote the perimeter,

$$\begin{aligned} p &= k + l + m + n \\ &= a\cos a + b\cos\beta + c\cos\gamma + d\cos\delta \\ &= a\sin\gamma' + b\sin\delta' + c\sin a' + d\sin\beta' \\ &= 2\mathrm{R}[\sin a'\sin\gamma' + \sin\beta'\sin\delta' + \sin\gamma'\sin a' + \sin\delta'\sin\beta'] \\ &= 4\mathrm{R}[\sin a'\sin\gamma'\sin\beta'\sin\delta'] \\ &= (ac + bd)/\mathrm{R} = hk/\mathrm{R}. \end{aligned}$$

GENERAL CASE.

In a similar way it may be shown that in any polygon of an odd number of sides, there may be inscribed a polygon of the same number of sides whose perimeter is a minimum; and that this inscribed polygon is uniquely determined. The directions of its sides and the positions of its vertices may in fact easily be calculated as in the case of the triangle.

In the case of a polygon with an even number of sides no polygon of minimum perimeter can be inscribed, unless a relation holds among the angles of the given polygon, namely, $A - B + C - D + \ldots$ must be zero. If this relation holds, an infinite number of polygons of minimum perimeter may be inscribed. Any two of these inscribed polygons have equal perimeters and have corresponding sides parallel.

PARTICULAR CASE.

The particular case of the regular polygon is of some interest.

The polygon of minimum perimeter inscribed in a regular odd polygon is a regular polygon having its vertices at the middle points of the sides of the given polygon.

Thus, in the case of the pentagon (fig. 35), putting $\angle APT = \angle BPQ = \alpha$, etc., we get, successively $\alpha = \gamma = \epsilon = \beta$; and therefore all the triangles, APT, BQP, etc., are isosceles. Also $PB = RD = DS = TA = AP$; and thus the vertices of PQRST bisect the sides of ABCDE.

If now we consider a regular hexagon (fig. 36), we do not get $\alpha = \beta$, by the reasoning that gave us this result in the case of the pentagon; but only $\alpha = \gamma = \epsilon$; and $\beta = \delta = \zeta$.

But if we suppose $\alpha > \beta$, we have

$$BQ > BP, \; CQ > CR, \; DS > DR,$$
$$ES > ET, \; FU > FT, \; AU > AP;$$

whence, adding, we have $BC + DE + FA > CD + EF + AB$; which is not the case; and therefore $\alpha = \beta$.

And in this case P is not necessarily the middle point of AB; for if we make $AP = CQ = CR = ES = ET = AU$, the hexagon PQRSTU will satisfy the conditions of the problem.

It may easily be shown that the perimeter of PQRSTU is constant; it is in fact $\sqrt{3}/2$ times that of ABCDE.

On some properties of the quadrilateral.

By R. E. ALLARDICE, M.A.

§ 1. In a triangle ABC (fig. 37), BE is made equal to CF; to find the locus of the middle point of EF.

Take K the middle point of BC and P the middle point of EF, then PK is the locus required. For if E′ and F′ are the middle points of BE and CF, the middle point of E′F′ will lie in PK (namely, at the middle point of PK); and again if BE′ and CF′ are bisected in E″ and F″, the middle point of E″F″ will lie in PK (namely, at the middle point of KR); and so on. At any stage we may double the parts cut off from BA and CA instead of bisecting them. Hence the locus required is such that any part of it, however small, contains an infinite number of collinear points; and hence the locus is a straight line.

§ 2. The proof of the above paragraph obviously holds good if BE and CF, instead of being taken equal, are taken in a constant ratio.

Hence the proposition may be stated as a property of the quad rilateral, as follows:—

If the points P and Q divide the sides AB and DC of a quadrilateral ABCD in the same (variable) ratio, the locus of the middle point of PQ is a straight line.

§ 3. GENERALISATION.

If L and M (fig. 38) divide the sides AB and DC of a quadrilateral ABCD in the same (variable) ratio $l{:}m$, then the locus of a point P that divides LM in a given ratio is a straight line.

For this is true, by last paragraph, if we take the ratio LP:LM to be 1:2; and hence also if we take it to be 1:4 or 1:8 or 3:8; or, in general, if we take it to be $m{:}2^n$; and hence it must be true generally.

Note.—In this way the whole plane may be divided into a number of quadrilaterals whose sides are proportional (but which are not similar).

§ 4. SECOND METHOD OF PROOF.

Lemma.—Let ABC, DEF (fig. 39) be two straight lines and DA,

EB, FC, be perpendicular to AC; and let further A′B′C′ (fig. 40) be a straight line and D′A′, E′B′, F′C′ be perpendiculars to A′C′, equal respectively to DA, EB, FC; then if AB:BC = A′B′:B′C′, the points D′, E′, F′, are collinear; and conversely.

The proof of this lemma comes at once on drawing through D and D′ lines parallel respectively to AC and A′C′.

[This lemma is obviously connected with a property of the simlpest kind of homogeneous strain.]

Now take (fig. 41) BF:FD = CG:GE. Bisect BC and DE in P and Q, and join PQ. Draw perpendiculars BH, CH′, etc., to PQ. Then, obviously, BH = CH′, DL = EL′; and HK:KL = H′K′:K′L′; and hence FK = GK′ and FR = RG.

The more general theorem of (§ 3) may also be proved in this way.

§ 5. Third Method of Proof.

As neither of the preceding proofs is exactly Euclidian in character, it may be as well to add the following proof.

Let ABC (fig. 42) be any triangle.

Make AC′ = AC; AB′ = AB.

Then BB′ is parallel to CC′, and P,Q,R,S, the middle points of BC, BC′, B′C, B′C′, are collinear.

Now BQ = CR = $\frac{1}{2}$ (BA + AC).

If we make BD = CE, we have still DQ = ER = $\frac{1}{2}$ (DA + AE); and hence RQ passes through P′, the middle point of DE.

The more general theorem, in which BD:CE is a constant ratio, may be proved in much the same way.

§ 6. By means of the proposition of the preceding paragraphs, a simple proof may be given of the following well-known theorem * :—

The middle points of the diagonals of a complete quadrilateral are concurrent. (Fig. 43).

Make BHDK a parallelogram.

$$\begin{aligned}\text{Then AH/HE} &= \text{(AH/HB)(HB/HE)}\\ &= \text{(BK/KF)(KC/KB)}\\ &= \text{KC/KF.}\end{aligned}$$

Hence the middle points of HK, AC and EF are collinear.

---

* *Numerous proofs of this theorem have been given, one of the simplest being that contained in Taylor's* Ancient and Modern Geometry of Conics, § 107. *For some account of the history of the theorem, see the last paper in this volume of the* Proceedings, *by Dr J. S Mackay.*

§ 7. The following proof, by means of co-ordinates, of the general theorem of (§ 3) is so simple, that it may be worth while giving it here.

Put (fig. 38) $\quad \mathrm{AL:LB = DM:MC} = \lambda:\mu$;
$\mathrm{AR:RD = LP:PM = BS:SC} = p:q$.

Let the co-ordinates of R, P, Q, be (ξ_1,η_1), (ξ_2,η_2), (ξ_3,η_3) ; the co-ordinates of A be (x_1,y_1), etc.

Then $\xi_1 = (qx_1 + px_4)/(p+q)$.

$$\xi_2 = \{p(\lambda x_3 + \mu x_4)/(\lambda+\mu) + q(\lambda x_2 + \mu x_1)/(\lambda+\mu)\}/(v+q)$$
$$= \{p(\lambda x_3 + \mu x_4) + q(\lambda x_2 + \mu x_1)\}/(\lambda+\mu)(p+q).$$
$$\xi_3 = (px_3 + qx_2)/(p+q).$$

Now we may easily show that if we put

$$\mathrm{P} = \{p(x_4 - x_3) + q(x_1 - x_2)\}/(\lambda+\mu)(p+q),$$
$$\mathrm{Q} = \{p(y_4 - y_3) + q(y_1 - y_2)\}/(\lambda+\mu)(p+q),$$

then $\quad \xi_2 - \xi_3 = \mu\mathrm{P}$; $\xi_3 - \xi_1 = -(\lambda+\mu)\mathrm{P}$; $\xi_1 - \xi_2 = \lambda\mathrm{P}$.

Hence
$$\eta_1(\xi_2 - \xi_3) + \eta_2(\xi_3 - \xi_1) + \eta_3(\xi_1 - \xi_2)$$
$$= \eta_1\mu\mathrm{P} - \eta_2(\lambda+\mu)\mathrm{P} + \eta_3\lambda\mathrm{P}$$
$$= \mathrm{P}\{\lambda(\eta_3 - \eta_2) + \mu(\eta_1 - \eta_2)\}$$
$$= \mathrm{P}\{\lambda(-\mu\mathrm{Q}) + \mu(\lambda\mathrm{Q})\}$$
$$= \mathrm{PQ}(-\lambda\mu + \lambda\mu) = 0.$$

Hence R, P and S are collinear.

An Apparatus of Professor Tait's was exhibited which gives the same curve as a glissette, either of a hyperbola or an ellipse.

Fourth Meeting, 28th February 1890.

R. E. Allardice, Esq., M.A., Vice-President, in the Chair.

On the Moduluses of Elasticity of an Elastic Solid according to Boscovich's Theory.

By Sir William Thomson.

The substance of this paper will be found in the *Proceedings of the Royal Society of Edinburgh*, Vol. xvi., pp. 693–724 ; and Thomson's *Mathematical and Physical Papers*, Vol. iii., Art. xcvii., pp. 395–498.

Fifth Meeting, 14th March 1890.

R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

On the different possible non-linear arrangements of eight men on a Chess-board.

By T. B. SPRAGUE, M.A.

The question having been proposed to me as a puzzle: To arrange eight men on a chess-board, so that no two of them shall be in the same line,—that is to say, that no two are to be in the same column, nor in the same row, nor in the same diagonal line,—I succeeded before very long in solving it by finding the annexed arrangement. (Fig. 45.)

Having been subsequently informed that other solutions had been found, I set to work systematically to ascertain how many there are, with the result that I found that there are twelve essentially different arrangements, and no more, which satisfy the required condition. The number is much larger, if we include the arrangements which are really the same, but are presented under a different aspect.

For instance, the above arrangement may be presented under eight aspects (Fig. 47). Here Nos. 2, 3 and 4 are got from No. 1 by turning the chess-board round clockwise, so that the side which was originally at the top becomes successively the right-hand side, the bottom side, and the left-hand side; each of the four sides of the board being uppermost in turn. No. 5 is got from No. 1 by inverting the position of the men on the board; and then Nos. 6, 7, 8 are got by turning it round as before.

It is clear that the number of arrangements which satisfy the condition that no two men shall be in the same column and no two in the same row, is 8.7.6.5.4.3.2; and if we assume that each essentially different arrangement (or, for brevity, each distinct arrangement) may be presented under eight aspects, we see that the total number of distinct arrangements which satisfy the condition, is 7.6.5.4.3.2, or 5040. The introduction of the further condition, that no two men are to be on the same diagonal line, reduces the number of arrangements to 12; but this is a conclusion

I have arrived at by trial, and I have not been able to prove mathematically that 12 is the correct number. If we start by putting a man on the square marked a in Fig. 46, then, by the conditions of the problem, the man in the second column cannot be placed on either of the three squares in it marked with a cross; and the man in the third column cannot be placed on either of the three squares in it similarly marked. We have thus five squares in the second column, on which the man can be placed; and if it is put on the square marked b, this prevents the man in the third column from being placed on either of the three squares marked with a circle; so that there are only two squares in the third column on which the man can be placed. If, however, we put the man on the square marked c in the second column, we shall find that this leaves four squares open in the third column; while, if we place the man on any one of the three remaining squares in the second column, there will be three possible positions in the third column. We see thus that the number of possible positions in any column depends on the squares occupied in the preceding columns, in a manner which does not seem to admit of mathematical treatment.

In order to make an exhaustive examination of all the possible arrangements, it is not necessary to examine all the 5040 mentioned above. Before describing the process, it will be desirable to adopt a notation, by means of which we may indicate any square on the board, and any arrangement of men. I number the columns from left to right, and the rows from top to bottom, as shown in Fig. 45; then, in referring to any square, the number of the column is placed first and the number of the row, second; thus the spaces occupied by the men in Fig. 45 will be denoted by (1,6), (2,1), (3,5), (4,2), (5,8), (6,3), (7,7), (8,4) respectively. When we wish to indicate the whole arrangement, it is unnecessary to write down the numbers of the columns, as these all run in regular succession; and the arrangement in Fig. 45 is sufficiently denoted by (61528374). Turning now to Fig. 47, we see that the eight arrangements are denoted by the following numbers:—

(1) 61528374	(5) $47382516 = i(1)$
(2) $57138642 = ip(1)$	(6) $42861357 = irp(1)$
(3) $52617483 = ir(1)$	(7) $38471625 = r(1)$
(4) $75316824 = rp(1)$	(8) $24683175 = p(1)$

On examining these, we observe that (5) is got from (1) by

inverting the order of the figures; and in the same way (6) is got from (4), (7) from (3), and (8) from (2). This process I call ***inversion***, and I denote it by the initial letter i, so that

$$i(61528374) = 47382516.$$

If we subtract each of the numbers in (1) from 9, we get 38471625, which is (7). This process I call *reversion*, and denote by r, so that

$$r(61528374) = 38471625.$$

Next, inverting the figures, we have

$$ir(61528374) = i(38471625) = 52617483,$$

which is (3). We have by these processes got the arrangements (5), (7), (3), from (1). In order to get the four remaining arrangements, we must interchange the columns and the rows. Thus in (1), (1,6) is changed to (6,1), (2,1) to (1,2), and so on; then arranging in the order of the new columns (or the old rows), we get (1,2), (2,4), (3,6), (4,8), (5,3), (6,1), (7,7) (8,5), or simply 24683175, which is No. (8). This process I call *perversion*, and denote by p, so that

$$p(61528374) = 24683175.$$

Then inversion of (8) gives us (2); reversion of (8) gives us (4); and inversion of (4) gives us (6); these processes being symbolically denoted as follows: *

$$ip(61528374) = i(24683175) = 57138642,$$
$$rp(61528374) = r(24683175) = 75316824,$$
$$irp(61528374) = i(75316824) = 42861357.$$

It will now be useful to show how, by means of the numbers, we

---

* It is not necessary for our present purposes to investigate the laws according to which our symbols of operation, i, r, p, combine with each other; and I will therefore content myself with stating a few of the principal laws, without any demonstration :—

$$i^2 = 1,\ r^2 = 1,\ p^2 = 1;$$
$$ir = ri,\ ip = pr,\ rp = pi;$$
$$irp = rip = ipi = rpr = pir = pri.$$

As an illustration of the use of these relations, let us take the processes by which arrangement (2) in Fig. 47 is got from No. (1); that is to say, the process of turning the chess-board clock-wise through a quadrant. We have seen that $(2) = ip(1)$, so that the operation will be denoted by ip. If, now, we repeat this process, we have $(ip)^2 = ip(ip) = ip.pr = ip^2r = ir$; and this, as we have seen, is the process by which (3) is got from (1). Again, if we repeat the same process once more, we have

$$(ip)^3 = ip(ip)^2 = ip.ir = rpr.r = rpr^2 = rp;$$

and this, as we have seen, is the process by which (4) was got from (1).

may ascertain, with regard to any particular arrangement, without any representation of it on a chess-board or in a diagram, whether it satisfies the condition that no two men are to be on the same diagonal line. For this purpose we need a sort of transformation of co-ordinates. If, instead of referring the position of each square to two sides of our board, as we have done hitherto, we refer it to one diagonal, AB (Fig. 48), and a perpendicular through its extremity, A, so as to fix the position of any square by the co-ordinates AM, PM, of its middle point, P, taking as our unit the half-diagonal of a square; then a square which is indicated by (xy) according to the former plan, will be indicated by $(x+y-1, x-y)$, according to the latter plan; or, if we produce the diagonal backwards, and take as our new origin a point O, distant half a diagonal from the corner, the square formerly denoted by (xy) will be now indicated simply by $(x+y, x-y)$.

According to this notation, the arrangement in Fig. 1 is indicated by (7,–5), (3,1), (8,–2), (6,2), (13,–3), (9,3), (14,0), (12,4). Here the fact that all the eight numbers which stand first in the pairs, are different,—7, 3, 8, 6, 13, 9, 14, 12,—shows that no two men are on the same diagonal line perpendicular to AB; and the fact that all the eight numbers which stand second in the pairs, are different, – 5, 1,—2, 2,—3, 3, 0, 4,—shows that no two men stand on the same diagonal line parallel to AB.

We see also that our problem may be stated without any mention of a chess-board: Required to arrange two sets of numbers, 1 - - - 8, in eight groups, each containing one out of each set, so that the sums of the numbers in the eight groups shall all be different, and the differences of those numbers shall also all be different.

As another illustration, I will show how, by the same operations i, r, p, all the aspects in the text, instead of being got from No. (1), are got from one of the others,—say (4).

We have $(4)=rp(1)$

$\therefore\ r(4)=r^2p\ (1)=p(1)$,

and $pr(4)=p^2(1)=(1)$; or $(1)=pr(4)$.

Then $(2)=ip(1)=ippr(4)=ip^2r(4)=ir(4)$;

$(3)=ir(1)=irpr(4)=i.ipi(4)=pi(4)$;

$(5)=i(1)=ipr(4)=i,ip(4)=p(4)$;

$(6)=irp(1)=irppr(4)=irp^2r(4)=ir^2(4)=i(4)$;

$(7)=r(1)=rpr(4)=rip(4)$;

$(8)=p(1)=ppr(4)=r(4)$.

It will be noticed that $(rp)^{-1}=pr$; and similarly $(ip)^{-1}=pi$, $(irp)^{-1}=pri=irp$.

I will now, by means of the preceding notation, explain the process of finding by trial the various possible arrangements. I begin by putting a man on the square (1,1); then I put a man in the second column on the highest admissible square, which is (2,3), because, by the conditions of the problem, the man must not be put on (2,1) or (2,2). In the third column, now, the man must not be put on (3,1), (3,2), (3,3), or (3,4): I therefore put it on (3,5). In the fourth column, the square (4,2) is admissible, and I therefore put the man there; and the man in the fifth column is similarly put on (5,4). Looking now at the position of the men on the board, we see that it is not possible, according to the conditions of the problem, to place a man anywhere in the sixth column. I therefore move on the last-placed man to the next admissible square, which is (5,8); but it is still not possible to place a man in the sixth column. I therefore remove the man from the fifth column, and move on the man in the fourth column to the next admissible square, namely (4,7). Then three more men may be placed, thus, (5,2), (6,4), (7,6); but it is not possible to place a man in the eighth column, and the last-placed men have therefore in succession to be moved on, and, when necessary, removed from the board. This explanation will, I trust, be sufficient to enable my readers, with the chess-board before them, to understand the process by which the following arrangements were successively arrived at and found inadmissible, until the last was arrived at, namely, 15863724.

13524 ×	13824 ×	1468253 ×	1526374 ×
——8 ×	——7 ×	——73 ×	1528374 ×
1357246 ×	13862 ×	146835 ×	——63 ×
——4 ×	——425 ×	——7 ×	——4 ×
1358246 ×	14253 ×	1473625 ×	157248 ×
——4 ×	——8 ×	1473825 ×	——63 ×
136275 ×	14273 ×	14752 ×	——8 ×
136824 ×	142837 ×	——82 ×	1582473 ×
—— 5 ×	——63 ×	1483 ×	——736 ×
137248 ×	1463 ×	14852 ×	15863724
——85 ×		——3 ×	

We can prove that this satisfies the conditions, by transforming it as above explained. First writing it in full, we get 1,1; 2,5; 3,8; 4,6; 5,3; 6,7; 7,2; 8,4; which for our present purpose may be more compactly written $\left|\begin{array}{c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 8 & 6 & 3 & 7 & 2 & 4 \end{array}\right|$. Writing the transfor-

mation in a similar way, we get $\left|\begin{array}{c|c|c|c|c|c|c|c} 2 & 7 & 11 & 10 & 8 & 13 & 9 & 12 \\ 0 & -3 & -5 & -2 & 2 & -1 & 5 & 4 \end{array}\right|$; and since the eight numbers in the upper line are all different, and the eight numbers in the lower line are also all different, we see that this is an arrangement which satisfies the conditions; or, briefly, a solution. Proceeding thus, we get the following twelve solutions:—

15863724	. . (1)	25741863	. . (5)	27581463	. (9)
16837425	. . (2)	26174835	. . (6)	*35281746	. (10)
24683175	. . (3)	26831475	. . (7)	35841726	. (11)
25713864	. . (4)	27368514	. . (8)	36258174	. (12)

As the work proceeds, various methods of shortening it slightly, suggest themselves; thus, if, when five columns, and consequently five rows, are occupied, the remaining three rows are adjacent rows, as, for instance, the fifth, sixth, and seventh; the problem is reduced to placing three men on a small board containing three squares in each side; and it is easy to see that this is impossible under the conditions. We therefore know that we cannot complete the arrangement, and it is useless to proceed to place a man on the sixth column. The same is the case if the two last remaining rows are adjacent.

We have to be on the watch against recording an old solution which presents itself under a new aspect. In order to explain this more clearly, I have given in the following table the numbers indicating four aspects of each of the twelve solutions.

		Reversion.	*Perversion.*	*Reversion of Perversion.*	
	a	$b=r(a)$	$c=p(a)$	$d=rp(a)$	
(1)	15863724	84136275	17582463	82417536	(1)
(2)	16837425	83162574	17468253	82531746	(2)
(3)	24683175	75316824	61528374	38471625	(3)
(4)	25713864	74286135	41582736	58417263	(4)
(5)	25741863	74258136	51842736	48157263	(5)
(6)	26174835	73825164	31758246	68241753	(6)
(7)	26831475	73168524	51468273	48531726	(7)
(8)	27368514	72631485	71386425	28613574	(8)
(9)	27581463	72418536	51863724	48136275	(9)
(10)	*35281746	*64718253	*53172864	*46827135	(10)
(11)	35841726	64158273	57142863	42857136	(11)
(12)	36258174	63741825	63184275	36815724	(12)

Here each aspect b is got from the corresponding a by reversion; each c is got from a by perversion; and each d is got from c by reversion. It is unnecessary to write down the remaining

four aspects of each solution, as they will be got by simple inversion of the four that are here given; but it will be convenient to denote them by A, B, C, D, respectively, so that $A = i(a)$, etc. This order is preferable in some respects to the order adopted in Fig. 47. We have placed (1)*a* first because it is the first solution that our process gives us. It will be noticed that solution (10) differs from all the others, in that we get no new aspects by inverting the order of the numbers in it; thus if we invert the order in (10)*a*, we get 64718253, which is (10)*b*; similarly the inversion of (10)*c* gives us (10)*d*. As an example of an old solution presenting itself under a new aspect, we may take 17582463, which the above table shows us is aspect *c* of solution (1). Having arrived at this solution in the course of our process, we see by inspection of the board that it is a new aspect of an old solution; for such inspection informs us that it is an aspect of a solution commencing with 15, which, therefore, we must have got already. It is easy to see further that, when in our trials we have got 17 as the commencing numbers, it is useless to make trial of arrangements that have a 2 in either the 3rd, 4th, 5th, or 6th place; because, if any one of these should give us a solution, it must be another aspect of one that we have already got. For a similar reason, if we begin with 14, it is useless to make trial of 142 . . . .; if we begin with 15, it is useless to make trial of 152 . . . . or of any arrangement containing 2 in the 4th place; if we begin with 16, we similarly reject arrangements with 2 in the 3rd, 4th, or 5th place; and when we begin with 18, we reject arrangements which have 2 in the 3rd, 4th, 5th, 6th, or 7th place.

There is no similar proposition when we begin with 2; but when we begin with 3, it is useless to try arrangements beginning with 31; for by interchanging the columns and the rows, each of those arrangements must give us an arrangement commencing with 2; and all such we have already had. Similarly, if we began with 4, it would be useless to try any arrangement which has 1 in the 2nd or 3rd column; if we began with 5, it would be useless to try any arrangement which has 1 in the 2nd, 3rd, or 4th column; and so on.

The new aspects which we have thus far considered, are those which are contained in the column *c*, and are obtained by the process I have termed perversion. In order to recognize the others, we must first see whether there is a man nearer to a corner square than the one we started with. For instance, our process leads us to the solution 35714286; and as the square (7,8) is next to a corner, this is

a solution we have already got, and our table shows us that it is the D aspect of No. (6). Even if a solution contains no man nearer to a corner than the one we started with, it may still be a new aspect of an old solution: for instance, 36824175 contains 2 other men that are at the same distance from a corner as the one we started with, namely, in the third square; and the arrangements commencing with these respectively are 35841726, which is No. (11); and 37285146, which is the B aspect of the same; while 36824175 is its C aspect. When we have completed all the trials that begin with 3, it is unnecessary to proceed any further; for, putting a man on the square (1,4), we see at once that, if no man is to be nearer to a corner than this man is, the only admissible arrangement of the four men on the extreme columns and rows is the one shown in Fig. 49; and on trial we find that only one other man can be placed on the board consistently with the conditions of the problem.

As regards the remaining four squares in the first column, we see that by commencing with any one of them we shall simply get the *b* aspects of the solutions we have already got; for instance, if we put a man on the 7th square, we shall get the *b* aspects of the solutions we got by putting the first man on the 2nd square.

On examining the twelve solutions we have got, the first thing that strikes us is the general absence of anything like symmetry, or apparent law, in them. It is, of course, clear that the conditions of the problem preclude bilateral symmetry; but we may have centric symmetry, so that to every man on the board there corresponds another at the same distance from the centre on the other side of it. One of our solutions is of this character, namely No. (10). In the numbers indicating it, 35281746, and 53172864, we observe that the sum of each pair of digits equidistant from the two ends, is 9; the 1st and the 8th, the 2nd and the 7th, and so on. It results from this centric symmetry that, as already mentioned, there are only 4 aspects of this solution, and not 8, as there are of the others. Hence the total number of arrangements which satisfy the conditions of the problem is $8 \times 11 + 4 = 92$; the total number of possible arrangements that contain one in each column and one in each row, being $8.7.6.5.4.3.2 = 40{,}320$.

From a study, either of the board or of the numbers in our table, we see that some of our solutions can be deduced the one from the other; for instance, if in No. (3)*a* we remove 2 from the first place

and put it in the last, we get 46831752,* which represents No. (4)A. In order to see whether a transformation of this kind is possible we may proceed as follows, referring the men to a diagonal of the board and its perpendicular, as previously explained.

Columns	x	-1,	0	1,	2,	3,	4,	5,	6,	7,	8	9,	10
Rows	y	7,	5	2,	4,	6,	8,	3,	1,	7,	5	2,	4
	$x+y$	6,	5	3,	6,	9,	12,	8,	7,	14,	13	11,	14
	$x-y$	-8,	-5	-1,	-2,	-3,	-4,	2,	5,	0,	3	7,	6

When we remove 2 from the first place to the last, and thus get 46831752, the new values of $x+y$ and $x-y$ are 11 and 7, which are different from any of the others; and this shows that the new arrangement is a solution. So, if we take 5 from the last place to the first,† we get a solution 52468317, which is (8)C. But if we attempt to repeat the process in either direction, we get no new solution. Taking, for instance, 75246831, we have a value of $x+y$, namely 6, which is the same as one of those we have already got; and this shows that the arrangement is not a solution. Similarly, if we take the arrangement 68317524, we get a value of $x+y$, namely 14, which we have already got; and this arrangement also is not a solution. I indicate this connection between the three solutions graphically as follows:—

× ——(8)C——(3)a——(4)A—— ×

Again, if we take the solution (11)c, namely 57142863, and subtract 1 from each of the numbers except 1, which is to be replaced by 8, we get 46831752, which is (4)A. This operation I denote by s (the initial letter of *subtract*), so that $46831752 = s(57142863)$, and $(4a) = is(11c)$. Each of these operations, s and t, has a simple interpretation with reference to the arrangement of the men on the chess-board, which is so easily understood when the men are placed on the board, that it is unnecessary to explain it. I represent the connection between (11)c and (4)A graphically thus:—

(11)c
|
(4)A

It will be found that seven of our solutions, namely, Nos. (1), (2), (3), (4), (6), (8), and (11), can be deduced one from the other by

* This operation I denote by t (the initial letter of *transpose*), so that $46831752 = t(24683175)$; whence $(4A) = t(3a)$.

† This operation will be denoted by t^{-1}, so that $52468317 = t^{-1}(24683175)$ and we have $(4A) = t(3a) = t^2(8C)$.

combinations of the s and t operations; and I think the reader with the board before him will have no difficulty in understanding, without further explanation, the following diagram, which represents the connection between them.

$$
\begin{tikzcd}
(8)C \arrow[r,dash] \arrow[d,dash] & (3)a \arrow[r,dash] \arrow[d,dash] & (4)A \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & (8)C \arrow[d,dash] \\
\times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[d,dash] \\
\times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[d,dash] \\
\times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[d,dash] \\
\times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & (8)D \arrow[r,dash] \arrow[d,dash] & (3)b \arrow[r,dash] \arrow[d,dash] & (4)b \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[d,dash] \\
\times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & (1)C \arrow[r,dash] \arrow[d,dash] & (6)C \arrow[r,dash] \arrow[d,dash] & (11)d \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[d,dash] \\
\times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & (2)d \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & (2)c \arrow[r,dash] \arrow[d,dash] & \times \arrow[d,dash] \\
(1)D \arrow[r,dash] \arrow[d,dash] & (6)D \arrow[r,dash] \arrow[d,dash] & (11)c \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & \times \arrow[r,dash] \arrow[d,dash] & (1)D \arrow[d,dash] \\
(8)C \arrow[r,dash] & (3)a \arrow[r,dash] & (4)A \arrow[r,dash] & \times \arrow[r,dash] & \times \arrow[r,dash] & \times \arrow[r,dash] & \times \arrow[r,dash] & \times \arrow[r,dash] & (8)C
\end{tikzcd}
$$

It will easily be seen that

$$
\begin{aligned}
(8C) = s(1D) = s^2t(2c) = s^3t^2(11d) &= s^4t^2(4b) \\
&= s^3t^3(6C) \\
&= \text{etc.}
\end{aligned}
$$

The remaining 5 solutions, Nos. (5), (7), (9), (10), and (12), do not admit of being transformed in a similar way. It will be noticed that (2)c and (2)d are derived from the adjacent arrangements in a different way from the rest, namely, by the compound operation st, or $s^{-1}t^{-1}$, or st^{-1}, or $s^{-1}t$. Thus (2)c is got from (1)D by carrying a man from the right hand bottom corner of the board to the left hand top corner, and moving all the other men diagonally through one square downwards to the right. These two solutions have each a

man in a corner square of the board, and I propose to call them "corner solutions".

Although I have satisfied myself by trial that there are 12 solutions and no more, I have not been able to discover any means of proving otherwise that this is the correct number. In order, if possible, to discover a law for the number of solutions, I have investigated the problem for the cases where the board has 4, 5, 6, 7, and 9 squares in its side; and the results are shown in the following table.

Number of Sides. (1)	Distinct Solutions. (2)	Of which are Centric. (3)	Total Solutions. (4)	Total Arrangements. (5)	Quotient (5)÷(4)
4	1	1	2	24	12
5	2	1	10	120	12
6	1	1	4	720	180
7	6	2	40	5,040	126
8	12	1	92	40,320	$438\frac{6}{23}$
9	46	4	352	362,880	$1030\frac{10}{11}$

There is no law visible in the progression of these numbers.

The solutions in the case of the boards containing 4, 5, 6, and 7 squares in a side, are given in Fig. 50.

The solutions (1), (2), (4), (6) of the 7-board can be derived one from the other in an endless series, as shown in the following scheme. Here I have not thought it necessary to indicate the different aspects of each solution, but the **1** printed in heavy type is a different aspect from the 1 printed in ordinary type.

1—6—6—**1**—2—4—2—1—6—6—**1**—2—4—2—1
| | | | | | | | | | | | | | |
1—2—4—2—1—6—6—**1**—2—4—2—1—6—6—**1**
| | | | | | | | | | | | | | |
2—1—6—6—**1**—2—4—2—1—6—6—**1**—2—4—2

The succession of the numbers vertically is easily seen to b
1—2—6—4—6—2—1—**1**.

The other solutions (3) and (5) cannot be similarly derived fr
... any other.

I. (1) 136824975
(2) —7285946
(3) —8692574
(4) 146392857
(5) ——825397
(6) —7382596
(7) ——925863
(8) —8397526
(9) 157938246
(10) ———42863
(11) —9642837
(12) 168374295
(13) 174835926
(14) ———9625

II. (1) 241796358
(2) —7139685
(3) —8396157
(4) —9731685
(5) ——53168*
(6) 257936418
(7) ——48136
(8) —8136974
(9) ——96374
(10) —-693147
(11) ————74
(12) —9418637
(13) 261379485
(14) —-753948*
(15) ——958473
(16) —3184975
(17) —9358417
(18) 275194683
(19) —9631485
(20) 281479635
(21) —5396417
(22) —6931475

III. (1) 358296174
(2) ——-7146
(3) —9247186
(4) 362951847*
(5) —8159247*
(6) —-519724
(7) —9741825
(8) 372859164
(9) 386192574

IV. (1) 427918536

Of these solutions, the four marked with a star are centric, namely II.(5), II.(14), III.(4), and III.(5); and it may be useful to give the graphic representations of them (*v.* Fig. 51).

In our search for a law, an obvious idea is to try whether any relation can be found to exist between the numbers of the solutions when the board has n squares in its side and when it has $n+1$. When we examine from this point of view the solutions we have found, we immediately see that it is only in exceptional cases that a solution in the one case can be deduced from one in the other case. If we take a solution for the board containing $(n+1)$ squares in its side, which for brevity we may call an $(n+1)$-board, and remove from the board one of the outside columns and one of the outside rows, we shall get an n-board, but we shall not in general get a solution; for, as there is a man on each column, and a man on each row, we shall have removed two men, unless it happens that there is a man on the corner square of the board which is common to the column and the row which we have removed. Conversely, when a solution for the n-board contains no man on one of the diagonals, we can get from this solution two solutions for the $(n+1)$-board, by producing the free diagonal, and adding a new column and a new row so as to intersect in the prolongation of the diagonal, first at the one end, and then at the other, and placing a new man on the corner square thus obtained. In this way solution (1) for the 5-board is got from

ɘ solution for the 4-board ; solutions (1) and (2) for the 7-board are ·t from the solution for the 6-board ; solutions (1) and (2) for the ·board are got from solution (3) for the 7-board ; and the following ɔlutions for the 9-board are got from solutions for the 8-board :—

Nos.	1, 6, 10, 14	for the 9-board from No.	4	for the 8-board.
„	7, 13,	„	„ 5	„
„	2, 5,	„	„ 6	„
„	8, 11,	„	„ 7	„
„	3, 12,	„	„ 9	„
„	4, 9,	„	„ 10	„

By examining the diagrams for the solutions for the 9-board, it is easy to see that they will give us 32 solutions for the 10-board. The numbers of these "corner solutions" are—

For the	4-board,		none
„	5	„	1
„	6	„	none
„	7	„	2
„	8	„	2
„	9	„	14
„	10	„	32

Here, again, there is no obvious law in the series of numbers.

It is sometimes possible to get a solution for the n-board from one for the $(n+1)$ board, by removing a man from the board, and both the column and the row containing it, and then closing up the board ; for instance, solution (2) for the 5-board gives in this way the solution for the 4-board. Conversely, one or more solutions for the $(n+1)$-board may sometimes be got by inserting a new column and a new row, and placing a man on the square in which they intersect. In this way we get :—

From Solution	(1)	for the 8-square,	1	solution for the 9-square.
„	(2)	„	1	„
„	(3)	„	0	„
„	(4)	„	5	„
„	(5)	„	3	„
„	(6)	„	2	„
„	(7)	„	4	„
„	(8)	„	0	„
„	(9)	„	3	„
„	(10)	„	3	„
„	(11)	„	1	„
„	(12)	„	1	„
		TOTAL,	24	

In consequence, however, of two of these solutions being given in 2 ways each, the actual solutions got are only 22, or 8 in addition to the corner solutions. There are thus 24 solutions for the 9-board which cannot be got from solutions for the 8-board. This method of treating the question therefore does not lead us to a law.

A study of the diagrams suggests several interesting propositions for investigation; for instance, in no one of the solutions we have got for the 6-, 7-, 8-, and 9-boards, is there an arrangement of 4 men such as we have them in the 4-board solution; and I am inclined to think it is impossible there should be, but I have not succeeded in proving it. Similarly, I am inclined to think it is impossible there should be an arrangement of 4 men on the outside columns and rows of a board, as shown in Fig. 52; but this also I have not succeeded in proving.

On the equations of Vortex motion, with special reference to the use of polar co-ordinates.

By C. Chree, M.A., King's College, Cambridge.

In several previous communications* to the Society, I have considered the equations of vortex motion in two dimensions in a compressible fluid. In the present communication I propose to consider certain forms of the hydro-dynamical equations of a more general kind. In certain cases the fluid will be supposed to be rotating, prior to the introduction of the vortex motion, with uniform angular velocity about a fixed axis.

Using the same notation as in my previous papers, and supposing the axes of x and y rotating with uniform angular velocity ω about the axis of z, which is supposed fixed, we can easily prove by the method of my paper "*On vortex motion in a rotating Fluid*" † that the equations of three dimensional motion are the following—

$$\frac{\delta u}{\delta t} - 2\omega v - \omega^2 x = -\frac{1}{\rho}\frac{dp}{dx} + X \quad \dots \quad \dots \quad (1),$$

$$\frac{\delta v}{\delta t} + 2\omega u - \omega^2 y = -\frac{1}{\rho}\frac{dp}{dy} + Y \quad \dots \quad \dots \quad (2),$$

$$\frac{\delta w}{\delta t} = -\frac{1}{\rho}\frac{dp}{dz} + Z \quad \dots \quad \dots \quad (3).$$

---

* *Proceedings*, vol. V., p. 52; vol. VI., p. 59; vol. VII., p. 29.

† *Proceedings*, vol. VII., p. 29.

The component velocities u and v are velocities relative to the moving axes. It will also be remembered that

$$\frac{\delta}{\delta t} \equiv \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz} \quad \ldots \quad \ldots \quad (4)$$

denotes differentiation following the fluid; while $\frac{d}{dt}$ denotes variation at a point fixed relative to the moving axes.

The equation of continuity is simply

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5).$$

The *form* of this equation is, it will be noticed, the same whether ω exist or not.

Employing the usual notation for the components of the *apparent** vorticity—

$$\xi = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ etc.}, \quad \ldots \quad \ldots \quad \ldots \quad (6),$$

we easily deduce from the preceding equations in the case when the components X, Y, Z of the external forces vanish or are derivable from a potential, the following equations † for the variation of the vorticity—

$$\rho\frac{\delta(\xi/\rho)}{\delta t} = \left[\xi\frac{d}{dx} + \eta\frac{d}{dy} + (\zeta+\omega)\frac{d}{dz}\right] u \quad \ldots \quad (7),$$

$$\rho\frac{\delta(\eta/\rho)}{\delta t} = \left[\xi\frac{d}{dx} + \eta\frac{d}{dy} + (\zeta+\omega)\frac{d}{dz}\right] v \quad \ldots \quad (8),$$

$$\rho\frac{\delta(\zeta+\omega)/\rho}{\delta t} = \left[\xi\frac{d}{dx} + \eta\frac{d}{dy} + (\zeta+\omega)\frac{d}{dz}\right] w \quad \ldots \quad (9).$$

In the left-hand side of (9) we have introduced $\frac{\delta\omega}{\delta t}$ which is of course zero. The components ξ, η are of course absolute as well as apparent vorticity components along the instantaneous directions of the corresponding axes. These equations will remain unaltered when we transfer the origin to any point fixed relative to the original rotating axes, the new axes of x and y being supposed to rotate with the same angular velocity ω as the original. In proof of this it is sufficient to remark that the co-ordinates of any point, whether fixed or moving, referred to the new axes will differ from those referred to

---

* See *Proceedings*, vol. VII., p. 32.

† Lamb's *Motion of Fluids*, Note D.

the old only by constant quantities independent of the time, and so all velocity and vorticity components are unaffected by the change.

The components of velocity and vorticity must satisfy (7), (8), (9) and in addition the ordinary boundary conditions and the equation), (5), of continuity. This last equation requires *inter alia* that two contiguous elements of fluid the one inside and the other outside a vortex filament have the same component of velocity along the normal to the surface of the filament.

As the fluid is supposed frictionless it is not necessary, unless it be involved in the equations (7) – (9), for the component velocities in elements just inside and just outside the surface of a vortex filament to be identical in any direction which lies in the tangent plane to the surface of the filament.

Suppose now that when ω is zero the components of velocity and vorticity in the fluid constituting a vortex ring, whose centre is in the axis of z and whose plane is parallel to xy, are given by

$$\left.\begin{array}{lll} u = \nu\cos\theta & v = \nu\sin\theta & w = w_0 \\ \xi = -\Omega\sin\theta & \eta = \Omega\cos\theta & \zeta = 0 \end{array}\right\} \quad \ldots \quad (10),$$

where $\theta = \tan^{-1}(y/x)$, and ν, w_0 and Ω are independent of θ.

These components will thus satisfy (7), (8), (9) and other necessary conditions when ω is zero. When, however, ω ceases to be zero the equations (7), (8), and (9) are no longer satisfied by (10). These equations may, however, be all satisfied by adding to the components (10) the additional terms

$$\left.\begin{array}{lll} u = \omega r\sin\theta & v = -\omega r\cos\theta & w = 0 \\ \xi = 0 & \eta = 0 & \zeta = -\omega \end{array}\right\} \quad \ldots \quad (11),$$

where $r^2 = x^2 + y^2$.

The additional terms in the velocity components add nothing to the components ξ and η of vorticity, and they are consistent with $\zeta = -\omega$.

To prove our statement we must substitute in (7), (8), and (9) the system of velocities and vorticities obtained by combining (10) and (11). Doing so and remembering that (9) is satisfied by the values (10) when $\omega = 0$, we find that it is still identically satisfied, and that (7) and (8) respectively lead to

$$-\sin\theta\frac{\delta\Omega}{\delta t} - \Omega\cos\theta\frac{\delta\theta}{\delta t} + \frac{\Omega\sin\theta}{\rho}\frac{\delta\rho}{\delta t} = \frac{\Omega}{r}\frac{d}{d\theta}(\nu\cos\theta + \omega r\sin\theta) \quad \ldots \quad (12),$$

$$\cos\theta\frac{\delta\Omega}{\delta t} - \Omega\sin\theta\frac{\delta\theta}{\delta t} - \frac{\Omega\cos\theta}{\rho}\frac{\delta\rho}{\delta t} = \frac{\Omega}{r}\frac{d}{d\theta}(\nu\sin\theta - \omega r\cos\theta) \quad \ldots \quad (13).$$

Multiplying (12) by $\cos\theta$, and (13) by $\sin\theta$, then adding and dividing out by Ω we simply get

$$\frac{\delta\theta}{\delta t} = -\omega.$$

This signifies that the fluid in the vortex ring moves relative to the rotating axes with angular velocity $-\omega$ about the axis of the ring, which is exactly the motion indicated by (11).

Again subtracting (12) multiplied by $\sin\theta$ from (13) multiplied by $\cos\theta$, we get

$$\frac{\delta\Omega}{\delta t} - \frac{\Omega}{\rho}\frac{\delta\rho}{\delta t} = \frac{\Omega\nu}{r},$$

which is precisely the same relation as when ω is zero.

The additional terms (11) in the velocity of the fluid possessed of vorticity have their resultant in the tangent plane at every point on the surface of the vortex ring, and so their existence in no way affects the equality of the velocity components inside and outside the ring in the direction of the normal to the surface.

This indicates that a circular vortex ring may exist in a rotating fluid with its plane perpendicular to the axis of rotation, provided it have, in addition to the motion existing in a similar ring in a non-rotating fluid, a uniform angular velocity about the axis of the ring which is equal in magnitude but opposite in direction to that of the undisturbed rotating fluid. As in the case of a non-rotating fluid, there may be an infinite plane boundary at right angles to the axis of rotation, the necessary conditions being satisfied by the existence of an image on the remote side of the plane.

If, for instance, we imagine the state of matters in the earth's atmosphere in latitude λ the same as in fluid limited by an infinite plane coincident with the tangent plane to the earth's surface, the fluid rotating with angular velocity $\omega\sin\lambda$ about an axis answering to the vertical at the point considered; then a vortex ring whose plane was horizontal would have an angular motion $\omega\sin\lambda$ in azimuth from east to west through south. In other words, a diameter of the ring connecting two definite material cross sections would appear to an observer at the point to rotate about the vertical in precisely the same way and at the same rate as the plane of vibration of a Foucault's pendulum.

Hitherto I have said nothing as to the motion in the fluid surrounding the vortex ring. When we consider this point a difficulty

appears which I do not see my way to answer satifactorily. It seems well worthy of notice.

The equations (7), (8), and (9) apply at every point in the fluid, and so must be satisfied by the fluid surrounding the vortex ring we have just considered. Now the velocity components due to the action of a vortex ring in the surrounding fluid satisfy the conditions of irrotational motion, but they depend both in magnitude and direction on the distance from the instantaneous position of the plane of the ring, Thus $\frac{du}{dz}$, $\frac{dv}{dz}$ and $\frac{dw}{dz}$ do not vanish in the fluid surrounding the ring, and so in accordance with (7), (8) and (9) unless $\zeta + \omega$ vanish everywhere, the whole of the fluid surrounding a vortex ring will take up some form of vorticity of a complicated character.

Theoretically as $\frac{du}{dz}$, etc., do not absolutely vanish, however distant the point considered from the vortex ring may be, in order to avoid setting up vorticity possessed of components ξ, η we require some species of motion set up whereby $\zeta = -\omega$ all through the fluid. This would require either the motion of the rotating fluid entirely to stop, or in addition to the rotation ω about the original axis a rotation $-\omega$ about some parallel axis. The two rotations would give a uniform translational velocity the same at every point occupied by the fluid.

In practice when we consider that every fluid is possessed of viscosity, and that the terms $\frac{du}{dz}$ etc., due to the action of an ordinary vortex ring become insensible at a small distance from the vortex, the conclusion we seem led to is that the vortex ring itself and the immediately adjacent fluid will possess, relative to the moving axes, a rotation about the axis of the ring in the direction opposite to that in which the undisturbed fluid rotates. Throughout the ring itself, if its vorticity be great, this angular velocity may approach indefinitely near $-\omega$, but it will gradually diminish in the external fluid as the distance from the ring increases and altogether vanish at distances where the direct action of the ring becomes insensible.

The previous remarks also indicate that any variation in any velocity component with the distance from a fixed plane perpendicular to the axis of rotation, whatever may be the exciting cause,

necessitates the existence of some form or other of apparent vortex motion.

For certain purposes it is desirable to employ polar co-ordinates in the hydrodynamical equations. Such equations are obtained by Basset in his *Treatise on Hydrodynamics.** His proof, however, is not of a very elementary character, and I think the following method shows more clearly the meaning to be attached to the symbols employed. For some of the results I shall refer the reader to a previous paper in the *Proceedings.*†

As in the paper referred to, the element of volume has three of its edges δr, $r\delta\theta$ and $r\sin\theta\delta\phi$ intersecting in P, the point whose co-ordinates referred to a fixed point 0, and the ordinary polar system are r, θ, ϕ. The first of those three line elements and the tangents to the other two at P form a system of orthogonal axes called the *fundamental* axes at P. If, as in the paper referred to, the corner of the element of volume opposite to P be termed S′, the co-ordinates of S′ are $r+\delta r$, $\theta+\delta\theta$, $\theta+\delta\phi$.

Also neglecting squares of the small quantities $\delta\theta$, $\delta\phi$, the cosines of the angles between the fundamental axes at S′ and those at P are given by the following scheme, reproduced from my previous paper,—

		at S′	
	r	θ	ϕ
at P r	1	$-\delta\theta$	$-\sin\theta\delta\phi$
at P θ	$\delta\theta$	1	$-\cos\theta\delta\phi$
at P ϕ	$\sin\theta\delta\phi$	$\cos\theta\delta\phi$	1

Let us first suppose that the axes are fixed, so as to avoid dealing with too many difficulties at once.

Let the velocity of a given element of fluid at the time t at the point P have the components u, v, w along the fundamental axes at P. After a short interval τ let this element be at S′, so that

$$\delta r = u\tau, \ \delta\theta = \frac{v}{r}\tau, \ \delta\phi = \frac{w}{r\sin\theta}\tau.$$

Then the velocity components at time $t+\tau$ along the fundamental axes at S′ are

$$u_1 = u + \frac{du}{dt}\tau + \frac{du}{dr}\delta r + \frac{du}{d\theta}\delta\theta + \frac{du}{d\phi}\delta\phi., \text{ etc.}$$

* Vol. II., Art. 470.

† Vol. III., p. 109.

Thus, referring to the above scheme, the components of the velocity of the element at time $t+\tau$, *along the fundamental axes at P* are

$$u_1 - v_1\delta\theta - w_1\sin\theta\delta\phi, \text{ etc.}$$

But the change per unit time in the component of the velocity in any given direction equals the accelerating force per unit mass in that direction. If then U, V, W be the components of the external forces per unit mass at the point P, proceeding to the limit when τ vanishes, we easily deduce for the equations of motion—

$$\frac{\delta u}{\delta t} - \frac{v^2+w^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} + \mathrm{U} \quad \ldots \quad \ldots \quad (14),$$

$$\frac{\delta v}{\delta t} + \frac{uv}{r} - \frac{w^2}{r}\cot\theta = -\frac{1}{\rho r}\frac{dp}{d\theta} + \mathrm{V} \ldots \quad \ldots \quad (15),$$

$$\frac{\delta w}{\delta t} + \frac{uw}{r} + \frac{vw}{r}\cot\theta = -\frac{1}{\rho r\sin\theta}\frac{dp}{d\phi} + \mathrm{W} \ldots \quad (16);$$

where

$$\frac{\delta}{\delta t} \equiv \frac{d}{dt} + u\frac{d}{dr} + \frac{v}{r}\frac{d}{d\theta} + \frac{w}{r\sin\theta}\frac{d}{d\phi} \quad \ldots \quad (17)$$

signifies as usual differentiation following the fluid.

These equations agree with Basset's, except that he has u instead of w in the last term on the left-hand side of (16). I believe, however, that the above is the correct equation.

By employing the method of flux it is very easily proved that the equation of continuity is

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t} + \frac{1}{r^2}\frac{d(ur^2)}{dr} + \frac{1}{r\sin\theta}\frac{d}{d\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{dw}{d\phi} = 0 \ \ldots \quad (18).$$

Let us next suppose that the entire system of fluid and axes is rotating with uniform angular velocity ω about the fixed axis $\theta = 0$.

Let u', v', w' denote the component velocities at time t at the point P (r, θ, ϕ) relative to the fundamental axes there, which are supposed fixed relative to the rotating axes; and let u, v, w be the absolute velocities in the direction of lines fixed in space with which the fundamental axes at P coincide at the time t. Then

$$u = u', \ v = v', \ w = w' + \omega r\sin\theta \quad \ldots \quad \ldots \quad (19).$$

Let R represent the position at time $t+\tau$, where τ is very small, of a point rigidly connected with the moving axes which at the time t coincided with the point P; then the components of the velocity at R at time $t+\tau$ relative to the fundamental axes there, which move with the fluid, are respectively

$$u' + \frac{du'}{dt}\tau, \quad v' + \frac{dv'}{dt}\tau, \quad w' + \frac{dw'}{dt}\tau;$$

and so the component velocities at R at time $t+\tau$ relative to fixed axes, coinciding with the instantaneous position of the fundamental axes there, are

$$u' + \frac{du'}{dt}\tau, \quad v' + \frac{dv'}{dt}\tau, \quad w' + \frac{dw'}{dt}\tau + \omega r \sin\theta \ldots \qquad (20).$$

For the last result it must be noticed that referred to axes fixed in space, coinciding with the position of the axes at O at time t, the r and θ co-ordinates of R and P are the same. From this and the consideration that the ϕ co-ordinate of R relative to the fixed axes exceeds that of P by $\omega\tau$, it also follows that the components (20) are identical with

$$u + \frac{du}{dt}\tau + \omega\tau\frac{du}{d\phi}, \quad v + \frac{dv}{dt}\tau + \omega\tau\frac{dv}{d\phi}, \quad w + \frac{dw}{dt}\tau + \omega\tau\frac{dw}{d\phi} \quad \ldots \qquad (21).$$

Thus, comparing (20) and (21) and remembering (19), we find

$$\frac{du'}{dt} = \frac{du}{dt} + \omega\frac{du}{d\phi}, \quad \frac{dv'}{dt} = \frac{dv}{dt} + \omega\frac{dv}{d\phi}, \quad \frac{dw'}{dt} = \frac{dw}{dt} + \omega\frac{dw}{d\phi}.$$

But from (19)

$$\frac{du}{d\phi} = \frac{du'}{d\phi}, \quad \frac{dv}{d\phi} = \frac{dv'}{d\phi}, \quad \frac{dw}{d\phi} = \frac{dw'}{d\phi};$$

and therefore

$$\frac{du}{dt} = \frac{du'}{dt} - \omega\frac{du'}{d\phi}, \quad \frac{dv}{dt} = \frac{dv'}{dt} - \omega\frac{dv'}{d\phi}, \quad \frac{dw}{dt} = \frac{dw'}{dt} - \omega\frac{dw'}{d\phi}.$$

Substituting these values and the values

$$\frac{du}{dr} = \frac{du'}{dr}, \qquad \frac{dv}{dr} = \frac{dv'}{dr}, \qquad \frac{dw}{dr} = \frac{dw'}{dr} + \omega\sin\theta,$$

$$\frac{du}{d\theta} = \frac{du'}{d\theta}, \qquad \frac{dv}{d\theta} = \frac{dv'}{d\theta}, \qquad \frac{dw}{d\theta} = \frac{dw'}{d\theta} + \omega r\cos\theta,$$

in the equations (14), (15) and (16) we get after reduction, dropping the dashes so that in the following equations u, v, w are the velocities *relative to the moving axes*,

$$\frac{\delta u}{\delta t} - \frac{v^2}{r} - \frac{(w+\omega r\sin\theta)^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} + \mathrm{U} \ldots \quad \ldots \quad \ldots \qquad (22),$$

$$\frac{\delta v}{\delta t} + \frac{uv}{r} - (w+\omega r\sin\theta)^2\frac{\cot\theta}{r} = -\frac{1}{\rho r}\frac{dp}{d\theta} + \mathrm{V} \quad \ldots \quad \ldots \qquad (23),$$

$$\frac{\delta w}{\delta t} + \frac{uw}{r} + \frac{vw\cot\theta}{r} + 2\omega(u\sin\theta + v\cos\theta) = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{dp}{d\phi} + \mathrm{W} \ldots (24).$$

In these equations $\frac{\delta}{\delta t}$ has the same *form* as in (17), and it also as there signifies differentiation following the fluid. The reader will, I think, find no great difficulty in this proof if he clearly realise that $\frac{du'}{dt}$ in (20) signifies the rate of change with the time of the component of the velocity relative to a certain moving axis, at a point whose co-ordinates are fixed relative to the moving axes; while $\frac{du}{dt}$ in (21) signifies the rate of change of the component of the velocity, relative to a certain axis fixed in space, at a point which is absolutely fixed in space.

As a simple illustration of these formulae, let us apply them to the case of the earth's atmosphere regarded as surrounding a rotating sphere of radius R, the acceleration due to whose attraction at distance r from its centre is $g\mathrm{R}^2/r^2$.

For the undisturbed condition, when the atmosphere is at rest relative to the sphere, we have $u=v=w=0$. Substituting these values along with $\mathrm{U}=-g\mathrm{R}^2/r^2$, $\mathrm{V}=\mathrm{W}=0$, we reduce the equations (22) - (24) to

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dr} &= \omega^2 r\sin^2\theta - g\mathrm{R}^2/r^2 \\ \frac{1}{\rho}\frac{dp}{d\theta} &= \omega^2 r^2\sin\theta\cos\theta \\ \frac{1}{\rho}\frac{dp}{d\phi} &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (25);$$

whose solution is, according as $p=k\rho$ or $p=k\rho^\gamma$,

$$\left.\begin{aligned} \log(\rho/\rho_0) &= \frac{\omega^2}{2k}r^2\sin^2\theta - \frac{g}{k}\mathrm{R}\left(1-\frac{\mathrm{R}}{r}\right) \\ \text{or} \qquad \rho^{\gamma-1}-\rho_0{}^{\gamma-1} &= \frac{\gamma-1}{2k\gamma}\left\{\omega^2 r^2\sin^2\theta - 2g\mathrm{R}\left(1-\frac{\mathrm{R}}{r}\right)\right\} \end{aligned}\right\} \quad (26).$$

In both results ρ_0 represents the density of the atmosphere at the surface of the earth at the poles.

These results may of course be obtained in a more elementary manner by the introduction of the idea of "centrifugal force," as in Besant's "*Hydromechanics.*"

In a paper* already referred to, I obtained the following expres-

---

* *Proceedings*, Vol. III., p. 113. *Cf.* Basset's *Treatise on Hydrodynamics*, Vol. I., Art. 18.

sions for the components of vorticity at the point (r, θ, ϕ) with respect to the fundamental axes there,

$$\left.\begin{aligned} \xi &= \frac{1}{2r\sin\theta}\left\{\frac{d}{d\theta}(w\sin\theta) - \frac{dv}{d\phi}\right\} \\ \eta &= \frac{1}{2r}\left\{\frac{1}{\sin\theta}\frac{du}{d\phi} - \frac{d}{dr}(wr)\right\} \\ \zeta &= \frac{1}{2r}\left\{\frac{d}{dr}(vr) - \frac{du}{d\theta}\right\} \end{aligned}\right\} \quad \ldots \quad (27).$$

When u, v, w are velocities referred to fixed axes, the above are components of the absolute vorticity, but when the velocities are relative to moving axes they are components of what I have called the *apparent* vorticity.

In order to apply our polar equations to vortex motion we require to obtain from (22), (23), (24) a series of equations corresponding to (7), (8), (9), when as there the external forces vanish or are derivable from a potential. The necessary algebraic operations are somewhat tedious and lengthy, so I shall merely indicate the method of procedure and give the results. The *verification* of the results will present no great difficulty to any one who keeps their form in view in grouping his terms, and who does not allow the length of his intermediate expressions to alarm him.

Replace in every case $\frac{\delta}{\delta t}$ by its equivalent in (17) and notice that, as the external forces have a potential,

$$U = \frac{dF}{dr}, \quad V = \frac{1}{r}\frac{dF}{d\theta}, \quad W = \frac{1}{r\sin\theta}\frac{dF}{d\phi},$$

where F is some function of r, θ, ϕ.

To obtain the first equation multiply (24) by $\sin\theta$, then differentiate with respect to θ, and from the result subtract (23) after differentiating it with respect to ϕ. Finally multiply up by $r/2\sin\theta$.

To obtain the second equation multiply (22) by $\operatorname{cosec}\theta$, then differentiate with respect to ϕ, and from the result subtract the result obtained by multiplying (24) by r and then differentiating it with respect to r. Finally divide out by $2r^2$.

To obtain the third equation multiply (23) by r, then differentiate with respect to r, and from the result subtract (22) differentiated with respect to θ. Finally divide out by $2r^2\sin\theta$.

In each case one of the objects of the preliminary operations is to eliminate p and the external forces.

In putting the resulting expressions into the following concise forms the identity (17), the equation of continuity (18), and the identities (27) are alone required. We thus finally obtain

$$\rho\,\frac{\delta}{\delta t}\left(\frac{r^2\xi}{\rho}\right)=\left[\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\right](r^2u)$$
$$-\frac{\omega}{\sin\theta}\left\{\frac{d}{d\theta}r\sin\theta(u\sin\theta+v\cos\theta)+r\cos\theta\frac{dw}{d\phi}\right\}\quad\ldots(28),$$

$$\rho\,\frac{\delta}{\delta t}\left(\frac{\eta}{r\rho}\right)=\left[\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\right]\left(\frac{v}{r}\right)$$
$$+\frac{\omega}{r^2\sin\theta}\left\{\frac{d}{dr}r\sin\theta(u\sin\theta+v\cos\theta)+\sin\theta\frac{dw}{d\phi}\right\}\ldots(29),$$

$$\rho\,\frac{\delta}{\delta t}\left(\frac{\zeta}{r\sin\theta\rho}\right)=\left[\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\right]\left(\frac{w}{r\sin\theta}\right)$$
$$+\omega\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right]\left(\frac{w}{r\sin\theta}\right)\ldots\quad\ldots\quad(30).$$

So long as ω is not zero these equations apply to the apparent not the absolute vorticities.

In the ordinary case when ω is zero and the axes are fixed, the following very concise form of the equations seems worth recording. Let Ω denote the resultant vorticity, now absolute, and ds an element in the direction of the axis of the resultant vorticity at any point in the fluid, then it is obvious that

$$\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\equiv\Omega\frac{d}{ds}\ldots\quad\ldots\quad(31);$$

and so the equations of vortex motion may be written

$$\rho\frac{\delta}{\delta t}\left(\frac{r^2\xi}{\rho}\right)=\Omega\frac{d}{ds}(r^2u)\quad\ldots\quad\ldots\quad(28a),$$

$$\rho\frac{\delta}{\delta t}\left(\frac{\eta}{r\rho}\right)=\Omega\frac{d}{ds}\left(\frac{v}{r}\right)\quad\ldots\quad\ldots\quad(29a),$$

$$\rho\frac{\delta}{\delta t}\left(\frac{\zeta}{r\sin\theta\rho}\right)=\Omega\frac{d}{ds}\left(\frac{w}{r\sin\theta}\right)\quad\ldots\quad(30a).$$

In employing polar co-ordinates it will also be convenient to possess equations answering to those in Cartesian co-ordinates of which in Lamb's* notation the type is

$$u=\frac{dP}{dx}+\frac{dN}{dy}-\frac{dM}{dz}.$$

---

* *Treatise on the Motion of Fluids*, Art. 129.

To transfer to polar co-ordinates let as usual

$$x^2+y^2+z^2\equiv r^2,\ z/r\equiv\cos\theta,\ y/x\equiv\tan\phi.$$

Then $$\frac{dr}{dx}=\sin\theta\cos\phi,\quad \frac{d\theta}{dx}=\frac{\cos\theta\cos\phi}{r},\quad \frac{d\phi}{dx}=-\frac{\sin\phi}{r\sin\theta},\quad \text{etc.}$$

Using these and taking the resultant along the three fundamental axes at each point of the components of the velocity in Cartesians, and denoting now the velocities along the fundamental axes by u, v, w, we easily find

$$u=\frac{dP}{dr}+\frac{1}{r}\left[\cos\phi\frac{d}{d\theta}-\sin\phi\cot\theta\frac{d}{d\phi}\right]M$$
$$-\frac{1}{r}\left[\sin\phi\frac{d}{d\theta}+\cos\phi\cot\theta\frac{d}{d\phi}\right]L+\frac{1}{r}\frac{dN}{d\phi},$$
$$v=\frac{1}{r}\frac{dP}{d\theta}+\left[\sin\phi\frac{d}{dr}+\frac{\cos\phi}{r}\frac{d}{d\phi}\right]L-\left[\cos\phi\frac{d}{dr}-\frac{\sin\phi}{r}\frac{d}{d\phi}\right]M$$
$$+\frac{\cot\theta}{r}\frac{dN}{d\phi},$$
$$w=\frac{1}{r\sin\theta}\frac{dP}{d\phi}+\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right]\left[L\cos\phi+M\sin\phi\right]$$
$$-\left[\sin\theta\frac{d}{dr}+\frac{\cos\theta}{r}\frac{d}{d\theta}\right]N.$$

Putting

$$\left.\begin{aligned}R&=\sin\theta(L\cos\phi+M\sin\phi)+N\cos\theta\\ S&=\cos\theta(L\cos\phi+M\sin\phi)-N\sin\theta\\ T&=M\cos\phi-L\sin\phi\end{aligned}\right\}\quad\ldots\quad(32$$

the above equations reduce to

$$\left.\begin{aligned}u&=\frac{dP}{dr}+\frac{1}{r^2\sin\theta}\left\{\frac{d}{d\theta}(Tr\sin\theta)-\frac{d}{d\phi}(Sr)\right\}\\ v&=\frac{1}{r}\frac{dP}{d\theta}+\frac{1}{r\sin\theta}\left\{\frac{dR}{d\phi}-\frac{d}{dr}(Tr\sin\theta)\right\}\\ w&=\frac{1}{r\sin\theta}\frac{dP}{d\phi}+\frac{1}{r}\left\{\frac{d(Sr)}{dr}-\frac{dR}{d\theta}\right\}\end{aligned}\right\}\quad\ldots\quad($$

If for shortness we represent the element of volume by dV employ the equation of continuity we can replace equation (1 Lamb's Art. 129 by

$$P=\frac{1}{4\pi}\iiint\frac{1}{\rho'}\frac{\delta\rho'}{\delta t}\frac{1}{r_1}dV'\quad\ldots\quad\ldots$$

a form applying to polar as well as Cartesian co-ordinates.

The integration is to be extended to every portion of space wherein the fluid is varying in density, and r_1 denotes the distance of the element $d\mathrm{V}'$ wherein the density is ρ' from the point where P is being calculated. This point of course is fixed and its co-ordinates constants so far as the above integration is concerned.

When it is desired to introduce polar co-ordinates explicitly into the above and subsequent integrals we know that

$$d\mathrm{V}' = r'^2 \sin\theta' dr' d\theta' d\phi'$$
$$r_1^2 = r^2 + r'^2 - 2rr'\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\}.$$

Lamb in the above-mentioned article also shows that if

$$\frac{d\mathrm{L}}{dx} + \frac{d\mathrm{M}}{dy} + \frac{d\mathrm{N}}{dz} = 0 \quad \ldots \quad \ldots \quad (35)$$

throughout the fluid, then L, M, N are like P expressible by integrals representing potentials. This condition holds if over every surface bounding the fluid the vorticity either vanishes or has its axis in the tangent plane to the surface. To represent these integrals in a form suitable for our present purpose, let Ω represent the resultant vorticity at any point and (PQ) the angle between any two directed quantities denoted by P and Q. Then the integrals are

$$\left.\begin{array}{l} \mathrm{L} = \dfrac{1}{2\pi}\displaystyle\iiint \frac{\Omega'\cos(\Omega' x)}{r_1} d\mathrm{V}' \\ \quad\text{etc.} \end{array}\right\} \quad \ldots \quad \ldots \quad (36).$$

We have now to find the equivalent in polar co-ordinates of equations (35) and (36).

Replacing differentiations with respect to x, y, z by differentiations with respect to r, θ, ϕ, and having regard to (32) after differentiation, we transform (35) into

$$\frac{1}{r^2}\frac{d}{dr}(\mathrm{R}r^2) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(\mathrm{S}\sin\theta) + \frac{1}{r\sin\theta}\frac{d\mathrm{T}}{d\phi} = 0 \quad \ldots \quad (37).$$

This result may also be obtained as follows. Suppose L, M, N to represent the components in the directions of the Cartesian axes of a certain vector quantity. Then under the conditions existing in (35) the vector quantity must satisfy the same condition as the velocity in an incompressible fluid. But by (32), R, S, T are the components of this same vector quantity along the fundamental directions at the point r, θ, ϕ. Thus they must satisfy the same condition as the velocity components in those directions in an in compressible fluid. Thus (18) should be satisfied when

$$\frac{\delta\rho}{\delta t}=0,\ u=\mathrm{R},\ v=\mathrm{S},\ w=\mathrm{T}\ ;$$

and when these substitutions are made we simply get equation (37).

We likewise easily find from (32) and (36), noticing that θ and ϕ are constants so far as the integrations are concerned,

$$\left.\begin{aligned}\mathrm{R}&=\frac{1}{2\pi}\iiint\frac{\Omega'\cos(\Omega'\xi)}{r_1}\,d\mathrm{V}',\\ \mathrm{S}&=\frac{1}{2\pi}\iiint\frac{\Omega'\cos(\Omega'\eta)}{r_1}\,d\mathrm{V}',\\ \mathrm{T}&=\frac{1}{2\pi}\iiint\frac{\Omega'\cos(\Omega'\zeta)}{r_1}\,d\mathrm{V}'\end{aligned}\right\}\quad\ldots\quad\ldots\quad(38).$$

If we denote by K the vector quantity whose components are L, M, N, or R, S, T, we obtain at once from (38)

$$\begin{aligned}\mathrm{R}\xi+\mathrm{S}\eta+\mathrm{T}\zeta&=\mathrm{K}\Omega\cos(\mathrm{K}\Omega)\\ &=\frac{1}{2\pi}\iiint\frac{\Omega\Omega'\cos(\Omega\Omega')}{r_1}\,d\mathrm{V}'\quad\ldots\quad\ldots\quad(39).\end{aligned}$$

Thus

$$\iiint\mathrm{K}\Omega\cos(\mathrm{K}\Omega)d\mathrm{V}=\frac{1}{2\pi}\iiiint\!\!\iint\frac{\Omega\Omega'\cos(\Omega\Omega')}{r_1}\,d\mathrm{V}d\mathrm{V}'\qquad(40).$$

The meaning of the sextuple integral is that the value of a certain function for the fluid occupying an element $d\mathrm{V}$ at the point $(r,\ \theta,\ \phi)$ is derived from an integration throughout space, and that the value of this function is then integrated throughout space.

When the fluid is incompressible and unlimited, being at rest at infinity, and all the vortices are within a finite distance of the origin, Lamb shows in his Art. 135 that a certain expression which is obviously identical with the right-hand side of (40) is equal to E/ρ, where E is the total kinetic energy of the fluid motion.

It may be seen from the same and previous articles that when the fluid, remaining incompressible, is not unlimited, the right-hand side of (40) still equals E/ρ provided over the boundary or boundaries, S, the axis of the resultant vorticity is in the tangent plane to the surface, and

$$\iint\mathrm{K}q\begin{vmatrix}\lambda & \mu & \nu\\ l & m & n\\ l' & m' & n'\end{vmatrix}d\mathrm{S}=0\quad\ldots\quad\ldots\quad(41);$$

where $(\lambda,\ \mu,\ \nu)$ are the direction cosines of K, $(l,\ m,\ n)$ the direction

cosines of the normal, and (l', m', n') the direction cosines of the resultant velocity q.

Supposing the above conditions satisfied when there exist a series of vortex filaments of strengths $m_1, \dots m_p, m_q \dots m_n$ forming closed curves, $s_1, \dots s_p, s_q \dots s_n$ in an incompressible fluid of density ρ, we obtain for the energy E of the fluid motion the equation

$$\mathrm{E} = \frac{\rho}{2\pi}\Sigma_1^n \left[m_p^2 \int\int \frac{\cos(ds_p ds'_p)}{r_{pp}} ds_p ds'_p \right] + \frac{\rho}{\pi}\Sigma\left[m_p\, m_q \int\int \frac{\cos(ds_p ds_q)}{r_{pq}} ds_p ds_q \right]$$

$$= \rho\ \Sigma_1^n \left[m_p \int \mathrm{K}\cos(\mathrm{K} ds_p) ds_p \right] \quad \dots \quad \dots \quad \dots \quad (42).$$

In the first summation ds'_p is an element of the same vortex filament as ds_p, and r_{pp} is the distance between them. In the second summation every pair of the n closed curves must be combined together, so that $\frac{1}{2}n(n-1)$ integrations are included. In the last summation K must of course be regarded as a function of the arc s_p of the curve along which the integration is being taken. The value of K depends on the combined action of the vorticity in the curve over which the integration is being taken and of the vorticity in each of the other $n-1$ closed curves. Basset gives in his Art. 92 formulae which are equivalent to the first of the two expressions given above for the energy; and points out the analogy of the first integrals to the co-efficients of self-induction and of the second integrals to the co-efficients of mutual induction of a series of closed electric currents.

The similarity of the equations of vorticity to those of electrodynamics * may, I think, be brought out very clearly by the following comparison of (39), (40), and (42) with corresponding electrodynamical expressions. We require first to notice that according to the table in Basset's p. 88, the following are analogous quantities in the two subjects :—

Vorticity Ω and electric current $\mathfrak{C}$,

K and electromagnetic momentum $\mathfrak{A}$.

Suppose with Maxwell in his *Electricity*, Art. 616, that the quantity he terms J is zero. Then treating equations (5) † of that

* *Cf.* Basset's Art. 95.

† Second Edition. In the first edition change μ into $1/\mu$ in these equations and in Art. 617.

article precisely as we treated our equations (38), we obtain as the equation corresponding to (39)

$$\mathfrak{A}\mathfrak{C}\cos(\mathfrak{A}\mathfrak{C}) = \mu \iiint \frac{\mathfrak{C}\mathfrak{C}'\cos(\mathfrak{C}\mathfrak{C}')}{r_1}\,d\mathrm{V}' \quad \ldots \quad (43).$$

The meaning of the dashes and of $d\mathrm{V}'$ and r_1 is exactly the same as in (39). The co-efficient μ is unity in the electromagnetic system and denotes the permeability in the electrostatic. In obtaining the equations from which (43) is derived Maxwell apparently treats μ as constant throughout every portion of space wherein currents are actually flowing.

From (43) we find the following equation answering to (40)

$$\iiint \mu^{-1}\mathfrak{A}\mathfrak{C}\cos(\mathfrak{A}\mathfrak{C})d\mathrm{V} = \iiiint\!\!\iint \frac{\mathfrak{C}\mathfrak{C}'\cos'(\mathfrak{C}\mathfrak{C}')}{r_1}\,d\mathrm{V}d\mathrm{V}' \ldots \quad (44),$$

the meaning of the sextuple integral being the same as previously.

Suppose now that the current system consists of currents of strengths $i_1 \ldots i_p, i_q \ldots i_n$ traversing closed linear conductors $s_1 \ldots s_p, s_q \ldots s_n$ of small cross section, then we may replace

$$\mathfrak{C}\ d\mathrm{V} \text{ by } i\ ds.$$

We thus transform (44) into

$$\tfrac{1}{2}\Sigma_1^n \left[i_p^2 \iint \frac{\cos(ds_p\, ds'_p)}{r_{pp}}\, ds_p\, ds'_p \right] + \Sigma \left[i_p\, i_q \iint \frac{\cos(ds_p\, ds_q)}{r_{pq}}\, ds_p\, ds_q \right]$$

$$= \tfrac{1}{2}\Sigma_1^n \left[i_p \int \mu^{-1}\mathfrak{A}\cos(\mathfrak{A}ds_p)\, ds_p \right] \quad \ldots \quad \ldots \quad (45).$$

The limits of integration and the meanings to be attached to the letters are the same as in (42). As $\mu = 1$ on the electromagnetic system the second side of (45) is, it will be seen, identical with the expression given in equation (14) of Maxwell's article 634 for the electrokinetic energy in the field. The first side is also a known form for the energy, the integrals of the first series being the co-efficients of self-induction, and those of the second series the co-efficients of mutual induction of the several circuits. The resemblance of (45) and (42) both in their final form and in their mode of derivation is very close.

I am not aware that any name has yet been assigned to the quantity K. It will appear, however, from the previous investigations that it is a quantity of considerable importance. It might with considerable fitness, as showing its analogy to the corresponding

electromagnetic quantity, be termed the *vector-potential of vorticity.* This name would also fit in with the mode of derivation from the vorticity, as the following geometrical interpretation of equations (38) will show.

We take the mean vorticity of each element of volume and divide it by the distance of the element from the point at which K is to be measured. Then starting from some fixed point as origin we draw, as in forming the polygon of forces, a series of lines representing these quantities in magnitude and in direction parallel to the corresponding vorticities. The line joining the origin to the end of the last line so drawn represents K in magnitude and direction.

This is practically the same construction as Maxwell gives for the corresponding quantity $\mathfrak{A}$ in his Art. 617. I fail, however, to see how he obtains the equation

$$\mathfrak{A} = \mu \iiint \frac{\mathfrak{C}}{r}\, dx\, dy\, dz$$

in the same article. It seems to me clearly wrong unless all the currents are parallel. Employing our previous notation, I believe the correct forms of this and of the corresponding hydrodynamical equation are

$$\mathfrak{A} = \mu \iiint \frac{\mathfrak{C}' \cos(\mathfrak{A}\mathfrak{C}')}{r_1}\, dV' \qquad \ldots \qquad \ldots \qquad (46),$$

$$K = \frac{1}{2\pi} \iiint \frac{\Omega' \cos(K\Omega')}{r_1}\, dV' \qquad \ldots \qquad \ldots \qquad (47).$$

In each case there exist two other equations determining the direction of the vector potential.

If in the equations (28*a*) – (30*a*) for a non-rotating fluid we make w, ξ and η vanish, and suppose along the vortex element r, u and v constants, we get the case of a circular vortex ring whose centre lies in the line $\theta = 0$ and whose plane is perpendicular to this line. The equations show that $\zeta/\rho r \sin\theta$ must be an absolute constant, and this may easily be seen to follow from the equation of continuity. Such a vortex ring, as I shall immediately show, can exist outside a sphere of which the line $\theta = 0$ is a diameter. Assuming this I shall first consider whether such a ring can exist when the fluid is rotating about the line in which the axis of the ring lies.

As in the similar case in Cartesian co-ordinates, it may easily be found that the equations (28) and (29) cannot be satisfied by sup-

posing w, ξ, η to vanish, and u and v to be independent of ϕ. Suppose, however, that when ω is zero these equations are satisfied by

$$\left.\begin{matrix} u=v_0 & v=v_0 & w=0 \\ \xi=0 & \eta=0 & \zeta=\zeta_0 \end{matrix}\right\} \quad \ldots \quad \ldots \quad (48.$$

Then when ω ceases to vanish they will be found satisfied by the following, which are consistent values,

$$\left.\begin{matrix} u=u_0 & v=v_0 & w=-\omega r\sin\theta \\ \xi=-\omega\cos\theta & \eta=\omega\sin\theta & \zeta=\zeta_0 \end{matrix}\right\} \quad \ldots \quad (49),$$

supposing ρ to have the same value as when ω vanishes.

The method of proof is the same in all three of equations (28) – (30), so it will be sufficient to exemplify it by treating (28) only.

Substitute the values (49) in (28), and then noticing that when ω is zero (49) satisfies the equation, we are left under the necessity of proving

$$\rho\frac{\delta}{\delta t}\left(-\frac{\omega r^2\cos\theta}{\rho}\right)=\omega\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right](r^2u_0)$$
$$+\frac{\omega r}{\sin\theta}\left\{\frac{d}{d\theta}\cdot\sin\theta(u_0\sin\theta+v_0\cos\theta)\right\}.$$

Substituting for $\frac{1}{\rho}\frac{\delta\rho}{\delta t}$ from (18), and then for $\frac{\delta}{\delta t}$ from (17), we have to prove

$$\left[u_0\frac{d}{dr}+\frac{v_0}{r}\frac{d}{d\theta}\right](\omega r^2\cos\theta)+\omega r^2\cos\theta\left\{\frac{1}{r^2}\frac{d(u_0r^2)}{dr}+\frac{1}{r\sin\theta}\frac{d}{d\theta}(v_0\sin\theta)\right\}$$
$$=\omega\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right](r^2u_0)+\frac{\omega r}{\sin\theta}\left\{\frac{d}{d\theta}\cdot\sin\theta(u_0\sin\theta+v_0\cos\theta)\right\}.$$

Carrying out the differentiations this will be found to be an identity. Similarly it may be shown that (49) satisfies (29) and (30). In the last case indeed the fact is obvious on inspection. This indicates that so far as the ring itself is concerned the equations may all be satisfied if the fluid in the ring possess, in addition to the velocity it would possess in a non-rotating fluid, a rotation with uniform angular velocity ω about the axis in an opposite direction to that in which the fluid and axes are rotating. Each material cross section of the ring in fact would move in a plane fixed in space.

The difficulties that occur in the motion of the fluid surrounding the ring are precisely the same as in the corresponding case in Cartesian co-ordinates. We may, as there, by bringing in the considera-

tion of the existence of viscosity in all actual fluids, arrive at a similar conclusion as to the probable character of the phenomena in nature.

In order to prove that the motion we have just investigated can apply to the case of a vortex ring when the fluid is bounded internally by a spherical surface, a diameter of which forms the axis of rotation and coincides with the axis of the vortex, we still require to show that all the requisite conditions for a vortex ring in this position can be satisfied when the fluid and axes are not rotating. All the data necessary for this already exist.

It has been shown by Mr T. C. Lewis* that if a vortex ring of strength m exist in an incompressible fluid outside a spherical boundary of radius a, with its core at a distance f from the centre, the boundary conditions are satisfied by the velocities deducible from the true ring and from an image ring of strength $-m\sqrt{f/f'}$ inside the sphere with its core at a distance $f' = a^2/f$ from the centre. Corresponding points of the two rings are, it will be noticed, in the same relative position as an electrified point and its image with respect to a spherical conductor.

Again it has been shown by Mr W. M. Hicks† that a "source" of strength m at a distance $f > a$ from the centre O of a spherical surface of radius a has an image inside the sphere, consisting of a "source" of strength ma/f at the point Q which is the inverse of the true source P, and of a line "sink" of strength m/a per unit length extending from Q to O.

Now by a source of strength m at P we simply mean that there exists a term $-m/r_1$ in the velocity potential, where r_1 is the distance from P. But from (34) if a vortex ring, whose element of cross section is $d\sigma$ and element of core ds, exist in a compressible fluid it contributes to P, which there denotes the velocity potential, the expression

$$\frac{1}{4\pi}\int\left\{\int\int\frac{1}{\rho'}\frac{\delta\rho'}{\delta t}\frac{1}{r_1}\,d\sigma\right\}ds.$$

The double integral inside the bracket extends over σ, and the integration with respect to ds is taken round the core.

Assuming, as usual, that the radius of the cross section is very small compared to b, the radius of the ring core, we may regard r_1 as

* *Quarterly Journal*, Vol. xvi., p. 338, or Basset's Art. 311, Vol. ii.

† *Phil. Trans.* 1880, Part ii., p, 455, or Basset's Art. 52, Vol. i.

constant over the cross section and equal to the distance of the external point from the core. We may then replace the above expression by

$$-\frac{1}{4\pi}\int\frac{\dot{\sigma}}{r_1}\,ds,$$

where

$$\dot{\sigma} = -\iint\frac{1}{\rho'}\frac{\delta\rho'}{\delta t}\,d\sigma.$$

I have shown on a previous occasion* that when ρ' is uniform over the cross section, b being very large compared to the radius of the section, we may take

$$\dot{\sigma} = \frac{\delta\sigma}{\delta t}.$$

Whether ρ' vary or not over the cross section, provided $\dot{\sigma}$ be the same for every point on the core, a ring of this kind is equivalent to a ring source of uniform strength $\dot{\sigma}/4\pi$ per unit length of core.

Applying Mr Hicks' result, denoting by a the radius of the sphere and by f the distance of the ring core from the centre, we find for the image a ring source at a distance a^2/f from the centre, whose strength per unit length of core is

$$\frac{a}{f}\frac{f^2}{a^2}\frac{\dot{\sigma}}{4\pi} = \frac{f}{a}\frac{\dot{\sigma}}{4\pi},$$

and a sink spread over the surface of a cone which extends from the image source to the centre. The strength of the annulus of the sink by two planes perpendicular to the axis of the cone intercepting unit length on the generators is everywhere

$$-\frac{2\pi b\dot{\sigma}}{4\pi a} = -\frac{b}{2a}\dot{\sigma}.$$

The strength of the cone sink per unit of surface would thus become infinite at its vertex, but that only means that there is a point sink of finite strength at the centre of the sphere. Combining the systems of images for the vorticity and for the compressibility, we clearly possess a solution of the problem whose solubility we had to establish.

While the preceding investigation as to the image system for the compressibility possesses the advantage of bringing out more clearly the physical side of the question than is possible in the analytical method, it would not in general lead very readily to the actual de-

---

* *Proceedings*, Vol. vi., p. 65.

termination of the fluid velocities. When the angle 2α of the cone subtended by the ring at the centre of the sphere is small, we could obtain approximate values for the velocity at points at a considerable distance from the ring by regarding it and the image ring as point sources and the image sink as a line sink.

In general recourse had better be had to the analytical expression for the velocity potential given below. Using a, b, α, and f in the same sense as above, we have $b = f\sin\alpha$.

Denoting by $Q_n(\theta)$ the nth zonal harmonic whose pole, $\theta = 0$, lies on the diameter of the sphere which is the axis of the ring, I find for the complete value of the velocity potential at the point (r, θ) referred to the centre of the sphere as origin

$$P = P_1 + P_2,$$

where

$$P_1 = -\frac{\dot{\sigma} a \sin\alpha}{2f}\left\{\frac{1}{2}\frac{a^2}{r^2}Q_1(\alpha)Q_1(\theta) + \dots \right.$$
$$\left. + \frac{n}{n+1}\frac{a^{2n}}{f^{n-1}r^{n+1}}Q_n(\alpha)Q_n(\theta) + \dots\right\} \quad \dots \quad (50),$$

$$\left.\begin{aligned} P_2 &= -\frac{\dot{\sigma}\sin\alpha}{2}\left\{1 + \frac{r}{f}Q_1(\alpha)Q_1(\theta) + \dots + \frac{r^n}{f^n}Q_n(\alpha)Q_n(\theta) + \dots\right\} \\ \text{or} \\ &= -\frac{\dot{\sigma} f\sin\alpha}{2}\left\{\frac{1}{r} + \frac{f}{r^2}Q_1(\alpha)Q_1(\theta) + \dots + \frac{f^n}{r^{n+1}}Q_n(\alpha)Q_n(\theta) + \dots\right\}\end{aligned}\right\} (51).$$

The first part of the velocity potential, P_1, may be regarded as answering to the image system. The second part P_2 is the velocity potential due to the ring itself; and the first or the second value for it is to be taken according as the point considered is nearer the centre of the sphere or is more remote than the ring core.

In calculating P_2 the method suggested in Thomson & Tait's *Natural Philosophy*, Art. 546, may be readily applied. The determination of P_1 is based on the vanishing of the normal component of the fluid velocity over the spherical surface. It presents no difficulty.

Over the spherical surface the velocity is everywhere of course along the tangent which lies in the plane through the axis of the vortex ring. At an angular distance θ from the axis its magnitude is

$$\left[\frac{1}{r}\left(\frac{dP_1}{d\theta} + \frac{dP_2}{d\theta}\right)\right]_{r=a} = \frac{\dot{\sigma}\sin\alpha\sin\theta}{2a}\left\{\frac{3a}{2f}Q_1(\alpha)Q_1'(\mu) + \dots \right.$$
$$\left. + \frac{2n+1}{n+1}\left(\frac{a}{f}\right)^n Q_n(\alpha)Q_n'(\mu) + \dots\right\} \quad \dots \quad (52),$$

where $Q_n'(\mu) = \frac{d}{d\mu} Q_n(\mu)$, denoting $\cos\theta$ by μ, and replacing the notation $Q_n(\theta)$ by $Q_n(\mu)$. The surface velocity thus vanishes, as is obvious from symmetry, where the axis of the ring cuts the spherical surface.

The expressions for the velocity potential in the form given above do not converge rapidly unless the distance of the ring from the centre of the sphere be considerable compared to the radius.

On a problem in permutations.

By R. E. Allardice, M.A.

The problem to be considered may be stated as follows :—How many necklaces may be formed with p pearls, r rubies, and d diamonds ?*

The peculiarity of this problem is that a general solution cannot be given in terms of p, r, and d alone. The form of the solution depends on the nature of the numbers p, r, and d; and it is necessary in solving the problem to consider whether or not these numbers have a common measure, and how many of them are odd and how many even. All possible cases of the problem are not discussed in this paper; but enough of them are considerod to illustrate the variety of forms that the solution may assume.

If we put $p+r+d=n$, the number of possible arrangements of the n stones in a line is $n!/p!r!d!$ Hence the question is, how many of these arrangements will give the same necklace; or, conversely, if we take any one form of the necklace, how many different arrangements of the stones we can get from it by breaking it at different parts and stretching it out straight. It is obvious that if the n stones had been all different, the answer to the second of these questions would have been $2n$; in other words, with n stones all different, we may form $n!/2n$ necklaces. The further question then naturally arises, In what cases, if any, are the $2n$ arrangements of the stones obtained from each form of the necklace all different when the stones are not all different? Now these $2n$ arrangements comprise the n that are obtained by a cyclical interchange of the stones, one at a time, together with the n that are obtained by exactly reversing each of these n arrangements.

---

* This problem was suggested to me by Professor Chrystal.

The $2n$ arrangements may fail to be all different for one of the three following reasons :—

(1). Because an arrangement is not altered when it is reversed. In this case the arrangement must be symmetrical.

(2). Because an arrangement is reproduced after a cyclical interchange of a certain number of the stones. It may easily be seen that in this case the arrangement considered must be resolvable into a number of identical groups.

(3). Because an arrangement is reproduced when a number of the stones have been interchanged cyclically and the resulting arrangement reversed. In order that this may happen, the arrangement considered must consist of two symmetrical groups. These three cases may be conveniently referred to as the case of a single symmetrical group, the case of identical groups, and the case of two symmetrical groups, respectively.

A single symmetrical group will occur if not more than one of the numbers p, r, d, is odd ; and not otherwise.

Identical groups will occur if p, r, and d have a common measure ; and not otherwise.

The case of the two symmetrical groups will occur if not more than two or the numbers p, r, d, are odd ; and not otherwise.

First Case.

Hence if p, r, and d are all odd and have no common measure, the number of necklaces that may be formed is $n!/2n.p!r!d!$, but this formula will not hold good in any other case. [It may be pointed out that any two of the numbers may have a common measure and the formula will still hold ; and that a similar formula will apply to the case of stones of any number of different kinds, provided there be an odd number of each of three of the kinds.]

Second Case.

Suppose now that two of the numbers, p, r, d, are odd ; but that these numbers have still no factor in common.

Let $p = 2\pi + 1$, $r = 2\rho + 1$, $d = 2\delta$.

We may now have two symmetrical groups, a pearl occurring at the centre of one and a ruby at the centre of the other. It should be noted further that a cyclical interchange in the case of two symmetrical groups produces two symmetrical groups ; and that by means of

a number of cyclical interchanges one of the groups may be reduced to a single stone. Hence, if we put $\nu = \pi + \rho + \delta$, the number of necklaces involving two symmetrical groups will be $\nu!/\pi!\rho!\delta!$. From each of these may be obtained n linear arrangements of the stones; and thus there are altogether $n!/p!r!d! - n.\ \nu!/\pi!\rho!\delta!$ arrangements that do not involve symmetrical groups, giving $(n!/p!r!d! - n.\nu!/\pi!\rho!\delta!)/2n$ necklaces. Hence the whole number of necklaces is

$$\frac{1}{2n}\left\{\frac{n!}{p!r!d!} - \frac{n.\nu!}{\pi!\rho!\delta!}\right\} + \frac{\nu!}{\pi!\rho!\delta!} = \frac{(n-1)!}{2.p!r!d!} + \frac{\nu!}{2.\pi!\rho!\delta!}$$

It may be well to illustrate this formula by actually writing out all the possible arrangements in a particular case.

Put $p=1$, $r=3$, $d=2$; then $\pi=0$, $\rho=1$, $\delta=1$; $n=6$, $\nu=2$.

The formula gives $\dfrac{5!}{2.1!3!2!} + \dfrac{2}{2.0!1!1!} = 6$.

The arrangements are the following, the two in the first column containing, and the others not containing, symmetrical groups:—

prdddr	*prrddd*	*pddrrd*
pdrdrd	*pddrdr*	*prddrd*

THIRD CASE.

Next let only one of the numbers p, r, d, be odd; and suppose that these three numbers have still no common measure.

Let $p = 2\pi + 1$, $r = 2\rho$, $d = 2\delta$, $n = p + r + d$, $\nu = \pi + \rho + \delta$. We have now the case of a single symmetrical group, and also the case of two symmetrical groups to consider.

In the latter case one of the groups will contain an odd number of stones and will have a pearl as the centre one, while the other group will contain an even number. It may easily be seen that any such arrangement may be reduced by a number of cyclical interchanges to a single symmetrical group.

The number of necklaces involving symmetrical groups is $\nu!/\pi!\rho!\delta!$, giving $n.\nu!/\pi!\rho!\delta!$ different linear arrangements. Hence it may easily be seen that the whole number of different necklaces is

$$(n-1)!/2.p!r!d! + \nu!/2.\pi!\rho!\delta!$$

This formula is exactly the same as that obtained in the last case; but π and ν have here slightly different meanings.

As an example, put $p=1$, $r=2$, $d=2$; so that $\pi=0$, $\rho=1$, $\delta=1$; $n=5$, $\nu=2$.

The formula gives $4!/2.1!2!2!+2!/2.0!1!1!=4$.

The arrangements are the following, the two in the first column containing, and the other two not containing, symmetrical groups :—

rdpdr	*rrpdd*
drprd	*drdrp*

FOURTH CASE.

Suppose in the next case, that p, r, and d are all even.

This case is an exception to the assumption made hitherto that p, r, and d have no common measure. I shall consider only the case in which the other factors, when 2 is divided out, are all odd numbers, and have no common measure, that is,

$p=2\pi$, $r=2\rho$, $d=2\delta$; where π, ρ and δ are odd numbers and have no common measure.

We may now have (1) a single symmetrical group; (2) two symmetrical groups; (3) identical groups.

Under (2) we have to consider two cases, namely, (α) that in which each of the symmetrical groups contains an odd number of stones, the central stones of the two groups being necessarily of the same kind; (β) that in which each of the symmetrical groups contain an even number of stones, an arrangement which is reducible to a single symmetrical group by means of cyclical interchanges.

(1). The number of necklaces involving single symmetrical groups is $\nu!/2.\pi!\rho!\delta!$, giving $n.\nu!/2.\pi!\rho!\delta!$ permutations.

(2). The number of necklaces involving two symmetrical groups of type (α) is $\Sigma\{(\nu-1)!/2.(\pi-1)!\rho!\delta!\}$, the three terms corresponding to the three cases where the middle stones are 2 pearls, 2 rubies and 2 diamonds. The number of permutations that these give is

$$n.\Sigma\{(\nu-1)!/2.(\pi-1)!\rho!\delta!\}=n.(\nu-1)!\Sigma\pi/2.\pi!\rho!\delta!=n.\nu!/2.\pi!\rho!\delta!$$

(3). The number of *permutations* involving identical groups is $\nu!/\pi!\rho!\delta!$ giving $(\nu-1)!/2.\pi!\rho!\delta!$ necklaces.

Hence the whole number of permutations not involving any of these three cases is

$$\frac{n!}{p!r!d!}-\frac{n.\nu!}{\pi!\rho!\delta!}-\frac{\nu!}{\pi!\rho!\delta!};$$

and the whole number of necklaces is

$$\frac{1}{2n}\left\{\frac{n!}{p!r!d!}-\frac{n.\nu!}{\pi!\rho!\delta!}-\frac{\nu!}{\pi!\rho!\delta!}\right\}+\frac{\nu!}{\pi!\rho!\delta!}+\frac{(\nu-1)!}{2.\pi!\rho!\delta!}$$

$$=\frac{(n-1)!}{2.p!r!d!}+\frac{\nu!}{2.\pi!\rho!\delta!}+\frac{(\nu-1)!}{4.\pi!\rho!\delta!}$$

As an example, put $p=2$, $r=2$, $d=2$; so that $\pi=1$, $\rho=1$, $\delta=1$, $n=6$, $\nu=3$.

The formula gives $5!/2.2.2.2+3!/2+2!/4=11$.

The arrangements are the following, those in (1) containing a single symmetrical group, those in (2) two symmetrical groups, those in (3) identical groups, and those in (4) having none of these peculiarities :—

$$(1)\ \left.\begin{matrix} dprrpd \\ pdrrdp \\ prddrp \end{matrix}\right\} \qquad (2)\ \left.\begin{matrix} pdrprd \\ drpdpr \\ rpdrdp \end{matrix}\right\} \qquad (4)\ \left.\begin{matrix} prddpr \\ rdpprd \\ pdrrpd \\ pprrdd \end{matrix}\right\}$$

$$(3)\ pdrpdr$$

Fifth Case.

Suppose next that p, r, and d have a factor in common. In a full discussion of the problem a number of particular cases would require to be considered; but I shall limit myself to that in which p, r, and d are all odd numbers and their G.C.M. is a prime number. In other words, I assume $p=\lambda\pi$, $r=\lambda\rho$, $d=\lambda\delta$, where λ, π, ρ, and δ are all odd numbers, λ is a prime, and π, ρ, and δ have no factor in common.

A permutation may consist of λ identical groups, each containing $\nu=\pi+\rho+\delta$ terms.

There will be $\nu!/\pi!\rho!\delta!$ such permutations, giving $(\nu!/\pi!\rho!\delta!)/2\nu$ necklaces.

Hence the whole number of necklaces is

$$\frac{1}{2n}\left\{\frac{n!}{p!r!d!}-\frac{\nu!}{\pi!\rho!\delta!}\right\}+\frac{(\nu-1)!}{2.\pi!\rho!\delta!}=\frac{(n-1)!}{2.p!r!d!}+\frac{(n-\nu).(\nu-1)!}{2n.\pi!\rho!\delta!}$$
$$=\frac{(n-1)!}{2.p!r!d!}+\frac{\lambda-1}{\lambda}\cdot\frac{(\nu-1)!}{2.\pi!\rho!\delta!}.$$

As an example, put $p=r=d=3$; so that $\pi=\rho=\delta=1$, $n=9$, $\nu=3$. The formula gives $8!/2.3!3!3!+2/3=94$.

Only one of the necklaces contains identical groups, namely, *prdprdprd*.

Without writing out all the 94 arrangements, we may verify that this is the correct number by enumerating them in the following manner :—

We may set down the three pearls in a ring, and count the number of ways in which the other six stones may be arranged relatively

to them. The number of stones to be put in each of the three spaces between the pairs of pearls is given in the following table, with the total number of corresponding necklaces in each case :—

0	0	6	gives	10	necklaces,
0	1	5	,,	20	,,
0	2	4	,,	20	,,
0	3	3	,,	10	,,
1	1	4	,,	10	,,
1	2	3	,,	20	,,
2	2	2	,,	4	,,
				94	

Sixth Meeting, 21*th April* 1890.

R. E. Allardice, Esq., M.A., Vice-President, in the Chair.

On a hydromechanical theorem.

By Dr A. C. Elliott.

Giffard's injector appeared more than thirty years ago. The first serious attempt to explain its action on dynamical principles was made by the late William Froude at the Oxford Meeting of the British Association in 1860. The history of mechanical science is almost everywhere deeply marked by Rankine; and it seems, just as it ought to be, that he should be found to have contributed not a little to the literature of this particular subject in a paper presented to the Royal Society of London in 1870. As serving to show how far the problem is still interesting, even from a high standpoint, attention may be directed to the exceedingly curious procedure of Professor Greenhill, where he deals cursorily with the matter at the page numbered 448 of his article on Hydromechanics in the *Encyclopædia Britannica.*

When first announced, the statement that the particles of a mere steam jet could, by the agency of this somewhat simple apparatus, force for themselves, in addition to a considerable quantity of more or less cold feed water, re-entrance into the identical boiler from whence they had escaped, seemed to involve an impossibility. But the mystery of that aforetime paradox would have been as nothing had it then been farther known what is now familiar—namely, that

the steam for the steam jet might be in some cases taken from the exhaust-pipe, instead of the boiler, with equally effective results.

In dealing with the theory of Giffard's injector, the author found that it was convenient to employ the principle of impulse and momentum; by which is meant simply a statement of Newton's second law in the form that the time-integral of the forces acting on the masses concerned, and reckoned in a specified direction, is equal to the resulting gain or destruction of momentum reckoned in the same direction.

In the ordinary symbols but merely altered in respect that f and v are regarded as component vectors, Newton's second law is represented by

$$f = m\frac{dv}{dt};$$

and hence

$$\int f dt = \int m dv.$$

The idea is applied in hydromechanics by considering a fluid-filled closed surface; and then the sum of the time-integrals of the external forces acting on or through the bounding surface is equal to the generation of momentum, all reckoned in one and the same direction. It is an instance of Mr Froude's remarkable insight that, in the paper referred to above, his explanation of Giffard's injector was based, so far as it went, on these principles. And it is worthy of remark, too, that Professor Greenhill, in the article already mentioned, by the help of Green's theorem, deduces Euler's equations directly from the fundamental principle.

But first we must obtain the equation of energy. Suppose a fluid to move steadily, irrotationally, and frictionlessly along a tube of stream lines. Let A_1, p_1, v_1 and ρ_1 be respectively the area, the pressure, the velocity, and the density at one section; and let similar symbols with the subscript $_2$ represent the same quantities for another and second section. Farther, let q be the volume corresponding to the mass-flow Q and density ρ. Then neglecting for our purpose differences of level, and equating the work done by the pressures on the end sections per unit time to the increase of energy, we have

$$p_1 A_1 v_1 - p_2 A_2 v_2 = Q\left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right) - \int_1^2 p dq \quad \ldots \quad \ldots \quad (1)$$

where the pressures are measured in terms of the weight of unit-mass. The equation of continuity gives

$$A_1 v_1 \rho_1 = A_2 v_2 \rho_2 = Q \; ; \qquad \ldots \qquad \ldots \qquad (2)$$

and

$$q\rho = Q. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

Also

$$\int p dq = pq - \int q dp. \qquad \ldots \qquad \ldots \qquad (4)$$

$\therefore$

$$\int_1^2 p dq = A_2 v_2 p_2 - A_1 v_1 p_1 - Q \int_1^2 \frac{dp}{\rho}. \qquad \ldots \qquad (5)$$

$\therefore$ (1) becomes

$$O = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} + \int_1^2 \frac{dp}{\rho} \; ; \qquad \ldots \qquad \ldots \qquad (6)$$

or

$$\frac{v^2}{2g} + \int \frac{dp}{\rho} = \text{const} = H. \qquad \ldots \qquad \ldots \qquad (7)$$

For the case of a liquid ρ is practically constant; and the equation of energy becomes

$$\frac{v^2}{2g} + \frac{p}{\rho} = H, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8)$$

where H may be called the total head.

From (8)

$$p = \rho \left\{ H - \frac{v^2}{2g} \right\}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9)$$

The equation of continuity gives

$$Av\rho = A_1 v_1 \rho = A_2 v_2 \rho = Q.$$

$\therefore$

$$p = \rho \left\{ H - \frac{1}{2g} \frac{Q^2}{\rho^2 A^2} \right\}. \qquad \ldots \qquad \ldots \qquad (10)$$

Multiplying by dA and integrating the impulse of the sides of the tube per second

$$= \int_1^2 p dA = \rho \left[H(A_2 - A_1) + \frac{Q^2}{\rho^2 2g} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \right]$$

$$= 2\rho H(A_2 - A_1) + p_1 A_1 - p_2 A_2. \ldots \qquad \ldots \qquad \ldots \qquad (11)$$

Now, considering the impulse on the areas A_1, A_2 it appears that, to obtain the total impulse, there falls to be added to the last expression

$$p_1 A_1 - p_2 A_2.$$

Hence the total impulse per second, say

$$J = 2\{p_1 A_1 - p_2 A_2 - \rho H(A_1 - A_2)\} \qquad \ldots \qquad \ldots \qquad (12)$$

$$= \frac{\rho}{g} \left\{ A_2 V_2^2 - A V_1^2 \right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (13)$$

$$=2\,(p_1-p_2)\frac{A_1A_2}{A_1+A_2}\,. \qquad \dots \qquad \dots \qquad \dots \qquad (14)$$

(13) may be obtained almost directly; for since impulse is equal to momentum generated

$$J=\frac{Q}{g}(v_2-v_1)$$

$$=\frac{\rho}{g}\left\{A_2v_2^{\,2}-A_1v_1^{\,2}\right\},$$

as before.

Take for an illustration of (14) the case of a liquid escaping from a large reservoir by a bell-mouthed tube into a space where the pressure is zero. Then $A_1=\infty$, $p_2=0$. Therefore

$$J=2p_1A_2, \qquad \dots \qquad \dots \qquad \dots \qquad (15)$$

and *not* simply p_1A_2, as might be hastily and erroneously concluded.

Secondly, suppose the fluid to be a gas and that the relation of pressure to density can be represented by

$$p=k\rho^{\gamma} \qquad \dots \qquad \dots \qquad \dots \qquad (16)$$

where k and γ are constants.

Reverting to (7) we have

$$\int\frac{dp}{\rho}=\frac{\gamma}{\gamma-1}\frac{p}{\rho}$$

$$=\lambda\frac{p}{\rho}, \qquad \dots \qquad \dots \qquad \dots \qquad (17)$$

where $\lambda=\dfrac{\gamma}{\gamma-1}$.

The gain of momentum is as before

$$J=\frac{Q}{g}(v_2-v_1);$$

and the equation of continuity gives

$$Q=Av\rho=\text{const.}$$

$$\therefore\ J=A_2\rho_2\frac{v_2^{\,2}}{g}-A_1\rho_1\frac{v_1^{\,2}}{g}. \qquad \dots \qquad \dots \qquad \dots \qquad (18)$$

Substituting from (17) in (6), and by the help of that result and (2) eliminating v_1 and v_2 from (18),

$$J=2\lambda\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)\frac{A_1\rho_1\,.\,A_2\rho_2}{A_1\rho_1+A_2\rho_2}. \qquad \dots \qquad \dots \qquad (19)$$

(14) and (19) are of course analogous; and the first may be obtained from the second by putting $\rho=\text{const.}$ The author has

found these expressions to possess some interest; and it was principally with the object of drawing attention to them that the present paper was entitled as above.

Let steam escape from a reservoir at pressure p_1 into a space at pressure p_2; and suppose the law

$$p = k\rho^{\gamma}$$

to hold. Then from (6)

$$\frac{v_2^2}{2g} = \int_2^1 \frac{dp}{\rho} = \lambda\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right); \quad \ldots \quad \ldots \quad (20)$$

$$\text{or,} \quad v = \sqrt{2g\lambda}\sqrt{\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}}.$$

Hence the mass-flow

$$Q = A_2\rho_2\sqrt{2g\lambda}\sqrt{\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}}. \quad \ldots \quad \ldots \quad (21)$$

Q is maximum when for a fixed value of p_1, p_2 has such a value that

$$\frac{p_1}{\rho_1}\rho_2^2 - p_2\rho_2 \text{ a maximum.}$$

That is when

$$\frac{p_2}{\rho_2} = \frac{2}{\gamma+1}\frac{p_1}{\rho_1}. \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Therefore

$$Q_{max.} = \rho_2 A_2\sqrt{2g\lambda}\sqrt{\frac{\gamma-1}{\gamma+1}}\sqrt{\frac{p_1}{\rho_1}}$$

$$= \rho_2 A_2\sqrt{2g}\sqrt{\frac{\gamma}{\gamma+1}}\sqrt{\frac{p_1}{\rho_1}}$$

$$= \sqrt{2g}\sqrt{\frac{\gamma}{\gamma+1}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} A_2\sqrt{p_1\rho_1}. \quad (23)$$

It is found by experiment that this maximum value of Q occurs; but that on continuously diminishing p_2 below the point corresponding to the maximum hardly any appreciable change takes place. In fact where $p_2 < \frac{3}{5}p_1$, an expression of the form (23) holds; and consequently

$$Q = mA_2\sqrt{p_1\rho_1} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where m is a constant not much differing from

$$\sqrt{2g}\sqrt{\frac{\gamma}{\gamma+1}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

Accepting the guidance of these facts, and reverting to (19), we find when $A_1 = \infty$ that

$$J = 2\lambda\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)A_2\rho_2. \quad \ldots \quad \ldots \quad (25)$$

This expression will be maximum for a given value of $\frac{p_1}{\rho_1}$ when

$$\frac{p_1}{\rho_1}\rho_2 - p_2$$

is maximum. That is when

$$\frac{p_2}{\rho_2} = \frac{1}{\gamma}\frac{p_1}{\rho_1}.$$

Hence the maximum value of J is

$$\begin{aligned} J_{max.} &= 2\lambda\frac{\gamma - 1}{\gamma}\frac{p_1}{\rho_1}A_2\rho_2 \\ &= 2\frac{p_1}{\rho_1}A_2\rho_2 \\ &= 2\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}p_1A_2 \quad \ldots \quad \ldots \quad (26) \\ &= 2np_1A_2 \quad \ldots \quad \ldots \quad \ldots \quad (27) \end{aligned}$$

where n is put for $\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$. We shall take this as a probable value of J for the steady state.

Fig. 55 represents diagramatically the essential parts of an injector. S is the steam pipe; W, the water inlet; O, the overflow; and B the pipe leading through the check valve to the boiler. Let p_1 be the boiler pressure; A_2 the area of the steam nozzle; and A_3 the area of the water-steam throat. The author intends on some future occasion to make an experimental determination of the pressure in the chamber where the steam and water mix. In the meantime, if the feed water be not too hot, we may assume that the pressure is but little greater than zero.

Let us suppose also that the water and steam pipes are large, and that the passages and nozzles are well and gently tapered. Then in applying (14), A_1 may be taken as $= \infty$. Farther, let v_3 represent the velocity of the water-steam jet, p_3 its pressure, and ρ the density; and put p_a for the atmospheric pressure (diminished by a quantity corresponding to the lift of the injector, if any). Then taking the sum of the impulses and equating to the gain of momentum, there results

$$2np_1A_2 + 2p_aA\cos a - p_3A_3 - f_1\rho\frac{v_3^2}{2g}A_3 = A_3v_3\rho \cdot v_3 \mid g, \quad \ldots \quad (28)$$

where f_1 is a friction and eddy resistance factor, A is the sectional area of the water passage projected on a plane at right angles to the stream lines, and a is the angle which a tangent to the stream lines at the same place makes with the axis of the injector. Now, from the equation of energy,

$$\frac{p_3}{\rho} = \frac{p_1}{\rho} - \frac{v_3^2}{2g} + f_0\frac{v_3^2}{2g};$$

or

$$\frac{p_3}{\rho} = \frac{p_1}{\rho} - f_2\frac{v_3^2}{2g} \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

where f_0 is a friction factor, and f_2 is written for $1 - f_0$.

Hence putting $A\cos a = sA_2$

$$v_3 = \sqrt{\frac{2g}{f_2}}\sqrt{\frac{2(np_1 + sp_a)A_2 - p_1A_3}{\rho(1 + f_1)A_3}}. \quad \ldots \quad (30)$$

The mass of steam used per second is say

$$S = mA_2\sqrt{p_1\rho_1};$$

and the mass of water passed on to the boiler per second is

$$A_3v_3\rho.$$

Therefore the mass of water injected per second, say

$$W = A_3v_3\rho - mA_2\sqrt{p_1\rho_1}. \quad \ldots \quad \ldots \quad (31)$$

Hence the ratio of the mass of water injected to the mass of steam used is

$$\frac{W}{S} = \frac{A_3v_3\rho}{mA_2\sqrt{p_1\rho_1}} - 1. \quad \ldots \quad \ldots \quad (32)$$

The efficiency of the apparatus will be maximum when

$$\frac{A_3v_3\rho}{mA_2\sqrt{p_1\rho_1}}$$

is maximum; that is when

$$\frac{A_3}{A_2}v_3$$

is maximum. Writing r for the ratio A_3/A_2, it appears from (30) that there is to be made maximum

$$2(np_1 + sp_a)r - pr^2.$$

Hence for best working

$$r = \frac{np_1 + sp_a}{p_1} \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

This value of r makes

$$v=\sqrt{\frac{2gp_1}{(1+f_1)\rho}}; \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

and

$$p_3=\frac{f_1}{1+f_1}p_1. \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

If f_1 were small enough, then this condition could not be fulfilled without

$$p_3<p_a$$

which is inadmissible with an open overflow.

On the other hand, if f_1 were large enough, this value for v could only be consistent with pressure above atmospheric in the overflow chamber, which, under the same conditions as formerly, is equally inadmissible.

It appears, therefore, that the best adjustment will tend towards the production of pressure in the overflow chamber above or below the atmosphere according as

$$\frac{f_1}{1+f_1}p_1 \text{ is } > \text{ or } < p_a.$$

With an open overflow we must have

$$f_2\frac{v_3^2}{2g}=\frac{p_1-p_a}{\rho}. \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

Therefore from (28) putting $p_3=p_a$,

$$\frac{p_1-p_a}{f_2\rho}=\frac{1}{\rho(2+f_1)}\left\{\frac{2}{r}(np_1+sp_a)-p_a\right\};$$

or

$$r=\frac{2(np_1+sp_a)}{\frac{2+f_1}{f_2}(p_1-p_a)+p_a}. \quad \ldots \quad \ldots \quad (37)$$

It appears also from (28) that in any case

$$r \text{ must be } < \frac{2(np_1+sp_a)}{p_1}. \quad \ldots \quad \ldots \quad (38)$$

If T_1 and T_f be respectively the temperatures of the steam and of the feed water in degrees Fahr., then in order that there might possibly just be condensation at the throat,

$$s=\frac{m\sqrt{p_1\rho_1}\{912+\cdot 3(T_1-32)\}}{\sqrt{2g\rho p_a}\{212-T_f\}}\cos\alpha;$$

but the mutual action is not actually so perfect, and it is well to allow

$$s=\frac{m\sqrt{p_1\rho_1}\{974+3(T_1-32)\}}{\sqrt{2g\rho p_a}\{150-T_f\}}\cos a. \quad \ldots \qquad (39)$$

Taking $\gamma=10/9$

$$n=\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}=\cdot 4 \text{ nearly.}$$

Suppose, further, for example, that the boiler pressure is 160 lbs. per sq. in. absolute; that the diameter of the water-steam throat is ·25 inch; and assume the following values:—

$$T_f=60°\text{F.};\ a=13°;\ f=\cdot 5;$$
$$f_2=\cdot 9;\ m=3\cdot 6, \text{ allowing for contraction.}$$

Then $r=42$; and diameter of steam nozzle

$$=\frac{\cdot 25}{\sqrt{\cdot 42}}=\cdot 38 \text{ inch.}$$

$$W=A_3\left\{\sqrt{2g\rho(p_1-p_a)/f_2}-\frac{m}{r}\sqrt{p_1\rho_1}\right\} \quad \ldots \qquad (40)$$

$$=3\cdot 0 \text{ lbs. per second.}$$

For the exhaust steam injector we have merely to substitute p_a for p_1 where it is associated with n, and in (31) where it appears under the radical. Thus

$$r=\frac{2p_a(n+s)}{\frac{2+f_1}{f_2}(p_1-p_a)+p_a} \quad \ldots \quad \ldots \qquad (41)$$

$$s=\frac{m\sqrt{\rho_a}\times 1028}{\sqrt{2g\rho}\{150°-T_f\}}; \quad \ldots \quad \ldots \qquad (42)$$

and

$$W=A_3\{\sqrt{2g\rho(p_1-p_a)/f_2}-\frac{m}{r}\sqrt{p_a\rho_a}\}\ldots \qquad (43)$$

On Rankine's Formula for Earth Pressure.

By Dr A. C. Elliott.

In a short course of lectures on "Railway Practice," the author was recently called upon to deal with the mechanical principles involved in the design of retaining walls. What has come to be known as Rankine's method had to be explained, at all events, in its

practical application. But the time at the author's disposal did not admit of the general consideration of the theory of stress by which Rankine in characteristic fashion leads up (in his *Applied Mechanics*) to the particular problem under discussion. The author had therefore to choose between omitting a demonstration or making a short cut, which at the same time should be of a rigorous nature.

The following process, which in the circumstances was the one employed, may be regarded, from a mathematical point of view, as a direct solution of the problem :—Given the angle between two conjugate stresses to determine their ratio when the maximum obliquity on *any* section has a given value. In the actual problem one conjugate stress is produced at any (a certain) point by the weight of the superincumbent column of earth ; the other conjugate stress is produced by the reaction of the wall ; and the maximum obliquity when the wall is exerting its least possible resistance is the angle of repose for the earthy material in question. The plane of rupture can of course be found, and it will be defined as that plane on which the obliquity of the stress has the given maximum value. (Fig. 56).

Let ABC represent a section of a triangular prism of earth whose length-axis is horizontal and parallel to the face of the wall ; and suppose moreover that the prism is of unit length. Also let $AB = a$ be vertical and parallel to the face of the wall ; and let BC be parallel to the upper surface of the earth. Let p_v be the intensity of the vertical thrust due to the weight of the earth ; and let p_h represent the conjugate thrust on AB. When, following Rankine, the friction of the wall is neglected the direction of p_h will be parallel to BC. Farther, let R be the total intensity of the stress on AC and α the obliquity ; β the angle which BC makes with the horizontal, and θ the angle BAC.

To find BC in terms of a we have

$$\frac{\mathrm{BC}}{a} = \frac{\sin\theta}{\cos(\theta+\beta)}.$$

Hence resolving vertically and using this value of BC

$$\mathrm{R}\sin(\theta+\alpha) = p_v a\frac{\sin\theta}{\cos(\theta+\beta)} - p_h a\sin\beta \; ; \qquad \ldots \qquad (1)$$

and resolving horizontally

$$\mathrm{R}\cos(\theta+\alpha) = p_h a\cos\beta. \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

$$\therefore \qquad \tan(\theta+\alpha) = \frac{1}{\mathrm{S}}\,\frac{\sin\theta}{\cos\beta\cos(\theta+\beta)} - \tan\beta, \quad \ldots \quad \ldots \qquad (3)$$

putting $$p_h/p_v = \mathrm{S}.$$

Now for a given value of S to find the plane AC for which a is maximum, put $\frac{da}{d\theta}=0$. Writing a explicitly

$$\theta + a = \tan^{-1}\left\{\frac{1}{\mathrm{S}}\frac{\sin\theta}{\cos\beta\cos(\theta+\beta)} - \tan\beta\right\}. \quad \ldots \quad (4)$$

$$\therefore \quad 1 = \frac{1}{1+\left\{\frac{1}{\mathrm{S}}\frac{\sin\theta}{\cos\beta\cos(\theta+\beta)} - \tan\beta\right\}^2}\cdot\frac{1}{\mathrm{S}\cos\beta}$$

$$\times\frac{\cos(\theta+\beta)\cos\theta + \sin\theta\sin(\theta+\beta)}{\cos^2(\theta+\beta)};$$

or
$$1 = \frac{1}{1+\left\{\frac{1}{\mathrm{S}}\frac{\sin\theta}{\cos\beta\cos(\theta+\beta)} - \tan\beta\right\}^2}\cdot\frac{1}{\mathrm{S}\cos^2(\theta+\beta)}. \quad \cdots \quad (5)$$

Equation (5) determines θ corresponding to a a maximum. When rupture is about to take place the maximum value of a is equal to the angle of repose, say ϕ. To find the value of S corresponding to this limiting state, we have therefore to eliminate θ between the two equations

$$\tan(\theta+\phi) = \frac{1}{\mathrm{S}}\frac{\sin\theta}{\cos\beta\cos(\theta+\beta)} - \tan\beta \quad \ldots \quad (6)$$

and
$$\mathrm{S}\cos^2(\theta+\beta) = \frac{1}{1+\left\{\frac{1}{\mathrm{S}}\frac{\sin\theta}{\cos\beta\cos(\theta+\beta)} - \tan\beta\right\}^2}. \quad \ldots \quad (7)$$

Using (6), (7) may be written

$$\mathrm{S}\cos^2(\theta+\beta) = \frac{1}{1+\tan^2(\theta+\phi)} = \cos^2(\theta+\phi); \quad \ldots \quad (8)$$

or
$$\sqrt{\mathrm{S}}\cos(\theta+\beta) = \cos(\theta+\phi). \quad \ldots \quad \ldots \quad (9)$$

Hence the two equations between which θ is to be eliminated may be written

$$\tan(\theta+\phi) = \frac{\sin\theta}{\sqrt{\mathrm{S}}\cos\beta\cos(\theta+\phi)} - \tan\beta, \quad \ldots \quad \ldots \quad (10)$$

and
$$\sqrt{\mathrm{S}}\cos(\theta+\beta) = \cos(\theta+\phi). \quad \ldots \quad \ldots \quad (11)$$

Writing in (10)

$$\sin\theta = \sin(\overline{\theta+\phi}-\phi),$$

expanding and transforming, there results

$$\tan(\theta+\phi)\left\{\frac{1}{\sqrt{S}}\frac{\cos\phi}{\cos\beta}-1\right\}=\frac{1}{\sqrt{S}}\frac{\sin\phi}{\cos\beta}+\tan\beta. \quad \ldots \quad (12)$$

Next putting (11) in the form

$$\sqrt{S}\cos(\theta+\phi-\overline{\phi-\beta})=\cos(\theta+\phi),$$

expanding and arranging, there results

$$\tan(\theta+\phi)=\frac{1}{\sin(\phi-\beta)}\left\{\frac{1}{\sqrt{S}}-\cos(\phi-\beta)\right\}. \quad \ldots \quad (13)$$

Combining (12) and (13)

$$\frac{1}{\sin(\phi-\beta)}\left\{\frac{1}{\sqrt{S}}-\cos(\phi-\beta)\right\}\left\{\frac{1}{\sqrt{S}}\frac{\cos\phi}{\cos\beta}-1\right\}=\frac{1}{\sqrt{S}}\frac{\sin\phi}{\cos\beta}+\tan\beta. \quad \ldots \quad \ldots \quad (14)$$

Multiplying up and reducing we have finally

$$S-2\sqrt{S}\frac{\cos\beta}{\cos\phi}+1=0. \quad \ldots \quad \ldots \quad (15)$$

$$\therefore \quad \sqrt{S}=\frac{\cos\beta}{\cos\phi}\pm\sqrt{\frac{\cos^2\beta}{\cos^2\phi}-1}. \quad \ldots \quad \ldots \quad (16)$$

$$\therefore \quad S=\frac{\cos\beta\mp\sqrt{\cos^2\beta-\cos^2\phi}}{\cos\beta\pm\sqrt{\cos^2\beta-\cos^2\phi}}. \quad \ldots \quad \ldots \quad (17)$$

And interpreting the signs according to the principle of least resistance, we have when the yielding is about to take place by failure of the wall

$$S=\frac{\cos\beta-\sqrt{\cos^2\beta-\cos^2\phi}}{\cos\beta+\sqrt{\cos^2\beta-\cos^2\phi}} \quad \ldots \quad \ldots \quad (18)$$

The other set of signs represents the case of yielding by *crushing in of the wall.*

[See also Elliott, *Proc.*, R.S.E., Jan. 1887.]

Seventh Meeting, 9th May, 1890.

R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

On a certain expression for a spherical harmonic, with some extensions.

By JOHN DOUGALL, M.A.

The object of this paper is to show how the leading properties of Spherical Harmonic Functions may be readily deduced by employing as a typical harmonic a certain simple algebraical expression, which obviously satisfies Laplace's equation ; and to extend a similar method to the case of any number of variables.

Some well-known expressions of Spherical Harmonics by means of Definite Integrals are readily arrived at by this method.

The most important properties of Spherical Harmonics, perhaps, are those connected with the integration of the product of two harmonics over the surface of a sphere, and it will be seen that this integral takes a somewhat remarkable form when the harmonics are of the type I have referred to.

Consider the function of x, y, z

$$u \equiv (ax + by + cz)^n$$

We have $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = (a^2 + b^2 + c^2)(ax + by + cz)^{n-2}$.

Hence if $a^2 + b^2 + c^2 = 0$, u is a harmonic of degree n.

Let $a = f + \iota f'$, $b = g + \iota g'$, $c = h + \iota h'$; $\iota = \sqrt{-1}$

The condition $a^2 + b^2 + c^2 = 0$ gives

$$f^2 + g^2 + h^2 = f'^2 + g'^2 + h'^2$$
$$ff' + gg' + hh' = 0$$

f, g, h ; f', g', h' being real.

Hence by transformation of rectangular axes, we may reduce u to the form $(x + \iota y)^n$, or, in polar co-ordinates, $r^n \sin^n\theta(\cos n\phi + \iota \sin n\phi)$; so that the real and imaginary parts of u are two conjugate sectorial harmonics. In what follows, then, we consider the general harmonic as made up of a sum of sectorial harmonics.

Observe that, by a well known proposition,

(1) $r^{2n+1}(ax+by+cz)^{-n-1}$ is another harmonic of degree n, where $r^2=x^2+y^2+z^2$.

Now let us consider the integral of the product of two harmonics of the type u, taken over the surface of a sphere of radius ρ.

If the degrees of the two harmonics are positive, but different, Green's Theorem shows at once that that integral is zero.

Let then $u_n \equiv (a_1x+b_1y+c_1z)^n$

$$v_n \equiv (a_2x+b_2y+c_2z)^n$$

n being a positive integer.

By Green's Theorem

$$\iint \frac{n}{\rho} u_n v_n dS$$

$$= \iiint \left(\frac{du_n}{dx}\frac{dv_n}{dx}+\frac{du_n}{dy}\frac{dv_n}{dy}+\frac{du_n}{dz}\frac{dv_n}{dz}\right)dV$$

and n being a positive integer, we may take the latter integral through the whole volume of the sphere.

But the function to be integrated through the volume being homogeneous of degree $2n-2$, we have

its volume integral = the product of its surface integral into

$$\int_0^\rho \frac{r^{2n-2}}{\rho^{2n-2}} \cdot \frac{r^2}{\rho^2} \cdot dr.$$

This last integral $= \dfrac{\rho}{2n+1}$.

Hence $\displaystyle\iint u_n v_n dS = \frac{\rho^2 n}{2n+1}(a_1a_2+b_1b_2+c_1c_2) \times \iint u_{n-1}v_{n-1}dS$

Continuing this method of reduction, we get finally

(2) $\displaystyle\iint u_n v_n dS = \frac{n(n-1)\ldots 1}{(2n+1)(2n-1)\ldots 3}.4\pi\rho^{2n+2}(a_1a_2+b_1b_2+c_1c_2)^n.$

And we may also write for reference

(3) $\displaystyle\iint u_n v_m dS = 0$ when, m, n are different.

In order to apply this result at once, let us find an expression for a Zonal Harmonic. We see that $(\iota\cos a.x+\iota\sin a.y+z)^n$ is a case of our form, and $\therefore \displaystyle\int_0^{2\pi}(\iota\cos.ax+\iota\sin ay+z)^n da$ is a harmonic.

By putting $x=\sqrt{x^2+y^2}\cos\phi$

$$y=\sqrt{x^2+y^2}\sin\phi$$

this becomes $\displaystyle\int_0^{2\pi}(\iota\sqrt{x^2+y^2}\cos\overline{a-\phi}+z)^n da$

$$=\int_0^{2\pi}(\iota\sqrt{x^2+y^2}\cos\beta+z)^n d\beta.$$

This function is symmetrical about Oz, and considering x, y, z as co-ordinates of a point on a sphere of radius unity, we see that when $x=0$ $y=0$ $z=1$, the value of the integral $=2\pi$.

Hence the zonal harmonic with axis Oz,

$$\text{say } \mathrm{P}_n(z)=\frac{1}{2\pi}\int_0^{2\pi}(\iota\sqrt{x^2+y^2}\cos\beta+z)^n d\beta.$$

Now our result (2) shows that in order to find the surface integral of the product of this and another harmonic $(ax+by+cz)^n$, we may substitute a, b, c for x, y, z, respectively in the expression for $\mathrm{P}_n(z)$ and multiply the result by a numerical coefficient which we may call k.

We thus get $\displaystyle\frac{k}{2\pi}\int_0^{2\pi}(\iota\sqrt{a^2+b^2}\cos\beta+c)^n d\beta.$

$$=\frac{k}{2\pi}c^n\int_0^{2\pi}(1-\cos\beta)^n d\beta$$

$$=\frac{4\pi}{2n+1}.c^n$$

Now c^n is the value of $(ax+by+cz)^n$ at the pole of $\mathrm{P}_n(z)$; and by adding together forms like $(ax+by+cz)^n$ we may get any harmonic.

Hence the well known important result which we may write

$$(4)\quad \iint \mathrm{P}_n\mathrm{V}_n dS=\frac{4\pi}{2n+1}\cdot\mathrm{V}'_n.$$

Now, considering further the form $(\iota x\cos a+\iota y\sin a+z)^n$, we see by Fourier's Theorem that if we expand it in cosines and sines of multiples of a, the coefficient of $\cos sa$ is $\displaystyle\frac{1}{\pi}\int_0^{2\pi}(\iota x\cos a+\iota y\sin a+z)^n$ $\cos sada$. This is obviously a harmonic, and by putting

$$x=\sqrt{x^2+y^2}\cos\phi$$
$$y=\sqrt{x_2+y^2}\sin\phi$$

and reducing, we see that it is a tesseral harmonic of type s.

The other of the same type is got by writing $\sin sa$ instead of $\cos sa$ in the integral.

Now consider the surface integral ($r=1$) of the product of the above harmonic under the $\int$ with another $\int_0^{2\pi}(\iota x\cos\beta + \iota y\sin\beta + z)^{n'}$ $\cos s'\beta d\beta$.

If n, n' are different, (3) shows that the result is zero. If $n'=n$ but s and s' different, we see writing out its value from (2) that the result is again zero. If $n'=n$, $s'=s$, the result is

$$k\int_0^{2\pi}\int_0^{2\pi}(1-\cos\overline{\alpha-\beta})^n\cos s\alpha\cos s\beta d\alpha d\beta.$$

We may expand $(1-\cos\overline{\alpha-\beta})^n$ in cosines of multiples of $\overline{\alpha-\beta}$, and the only term which contributes anything to the integral is that containing $\cos s(\alpha-\beta)$.

Expanding by ordinary trigonometry, and integrating, we get

$$(5)\ \frac{k}{2^{n-1}}(-1)^s\pi^2\frac{2n!}{(n-s)!(n+s)!}=(-1)^s\frac{(2\pi)^3 n!n!}{(2n+1)(n-s)!(n+s)!}$$

In order to obtain expansions of harmonics in polar co-ordinates,

$$\text{put } \left.\begin{aligned}\xi&=x+\iota y\\ \eta&=x-\iota y\end{aligned}\right\}\quad (x^2+y^2+z^2=1).$$

$(ax+by+cz)^n$ becomes $\left(\frac{a(\xi+\eta)}{2}+\frac{b(\xi-\eta)}{2\iota}+cz\right)^n$

$$=\left(\frac{\iota c}{2}\right)^n(a\xi+\frac{1}{a}\eta-2\iota z)^n \text{ say.}$$

Now taking $a\xi+\frac{1}{a}\eta$ as a single term, we may expand this by the Binomial Theorem, and pick out those terms in which the power indices of ξ and η differ by s.

Doing this we get $(a\xi+\frac{1}{a}\eta-2\iota z)^n$

$$=\Sigma\{(ae^{\iota\phi})^s+(ae^{\iota\phi})^{-s}\}\frac{n!}{s!(n-s)!}(-2\iota)^{n-s}\theta_n^{(s)}$$

$\theta_n^{(s)}$ being the function given in Thomson and Tait's Natural Philosophy, Vol. I., p. 205. Taking as a particular case the form $(\iota x\cos\alpha + \iota y\sin\alpha + z)^n$, we easily find from this

$$\int_0^{2\pi}\overline{(\iota x\cos\alpha+y\sin\alpha+z)}^n\cos s\alpha d\alpha=\frac{\iota^{2n-s}}{2^{s-1}}\frac{n!}{s!(n-s)!}\theta_n^{(s)}\cos s\phi$$

Another mode of expansion leads to other well-known forms. Observe $(a\xi+\frac{1}{a}\eta-2\iota z)^n$ may be written in either of the forms

$$\frac{1}{(a\xi)^n}\left\{1-(z+\iota a\xi)^2\right\}^n$$

$$\left(\frac{a}{\eta}\right)^n\left\{1-\left(z+\frac{\iota\eta}{a}\right)^2\right\}^n$$

Expanding these in powers of a by Taylor's Theorem, we get the expressions for the harmonics in terms of differential co-efficients of $(1-z^2)^n$.

The expansion of the Biaxal surface Harmonic may be deduced at once from (4) and (5).

The definite integral expressions for the elementary harmonics are particular cases of expressions as the sum of a finite number of terms.

Thus let $V=(\iota x\cos a+\iota y\sin a+z)^n=\frac{1}{2}H_0+H_1\cos a+H_2\cos 2a+\ldots$
$+K_1\sin a+K_2\sin 2a+\ldots$

and take a_1, a_2, ... a_p a series of angles in equi-different progression, the common difference being $2\pi/p$.

Then $V_1=(\iota x\cos a_1+\iota y\sin a_1+z)^n=\frac{1}{2}H_0+H_1\cos a_1+H_2\cos a_2+$ &c., and it is easy to show that

$$V_1\cos sa_1+V_2\cos a_2+\ldots\ldots V_p\cos sa_p=\frac{p}{2}H_s.$$

the other terms on the right hand side vanishing, provided p is not less than $2n+1$.

In this way we may express any harmonic as a sum of sectorial harmonics.

In particular,

$$V_1+V_2+\ldots V_p=\tfrac{1}{2}pH_0.$$

Taking $p=\infty$, we get the definite integral

$$\frac{1}{2\pi}\int_0^{2\pi}(\iota x\cos a+\iota y\sin a+z)^n da$$

as before.

We may interpret this integral in a way that will lead us naturally to one or two others of similar form.

Suppose we have a function of x, y, z, the variables being connected by a relation, say $x^2+y^2+z^2=\rho^2$; so that to any particular value of z correspond in general an infinite system of values of x, y, say x_1y_1, x_2y_2, &c.

Then, obviously, the sum of *all* the values which the function takes for this particular value of z, say $f(x_1, y_1, z)+f(x_2, y_2, z)+$ &c.

(multiplied by a suitable infinitesmal) is a function of z and ρ alone. In particular cases, of course, it may be zero, or infinite.

To treat the matter perfectly generally, we should have to consider x, y, z as complex variables, but we may get a sufficient view under the following restrictions, viz. : we consider z as real, but x and y capable of taking values either *purely* real or purely imaginary.

We may consider surface values, *i.e.*, suppose $x^2+y^2+z^2=1$.

Take then the harmonics $(z+\iota x)^n$, $(z+\iota x)^{-n-1}$, both of which by (1) give us surface harmonics of the n^{th} degree, and we get the following cases :—

I. $z<1$

(i) $x=\sqrt{1-z^2}\cos a$

$y=\sqrt{1-z^2}\sin a$

Harmonics are $\int_0^{2\pi}(z+\iota\sqrt{1-z^2}\cos a)^n da$

$\int_0^{2\pi}(z+\iota\sqrt{1-z^2}\cos a)^{-n-1}da$

(ii) $x=\sqrt{1-z^2}\cosh a$

$y=\sqrt{1-z^2}\,\iota\sinh a$

Harmonic $\int_{-\infty}^{+\infty}(z+\iota\sqrt{1-z^2}\cosh a)^{-n-1}da$

II. $z>1$

(i) $x=\iota\sqrt{z^2-1}\cos a$

$y=\iota\sqrt{z^2-1}\sin a$

Harmonic $\int_0^{2\pi}(z-\sqrt{z^2-1}\cos a)^n da$

$\int_0^{2\pi}(z-\sqrt{z^2-1}\cos a)^{-n-1}da$

(ii) $x=\iota\sqrt{z^2-1}\cosh a$

$y=\sqrt{z^2-1}\sinh a$

Harmonic $\int_{-\infty}^{+\infty}(z-\sqrt{z^2-1}\cosh a)^{-n-1}da.$

These are well known expressions for Zonal Harmonics of the first and second kinds. By similar considerations we might also obtain integrals for the harmonics which are not zonal.

We go on to sketch briefly the application of the preceding methods to functions of p variables.

The function $v \equiv (a_1x_1 + a_2x_2 + \ldots a_px_p)^n$ satisfies the equation $\frac{d^2v}{dx_1^2} + \frac{d^2v}{dx_2^2} + \ldots + \frac{d^2v}{dx_p^2} = 0$, provided $a_1^2 + a_2^2 + \ldots + a_p^2 = 0$.

The fundamental proposition about the integral of the product of two harmonics can be extended to the general case. To avoid circumlocution, we speak of points, lines, surfaces, volumes, by a well understood extension of the language of Analytical Geometry.

Now we may extend Green's Theorem to the case of p variables, the proof following exactly the same lines as in the case of three variables. The only difficulty is about the meaning of the element of surface, but we may define it in this way.

Let the surface be $F(x_1, x_2 \ldots) = 0$, and transform the variables linearly and orthogonally to a new set ξ_1, ξ_2, &c., of which $\xi_1 = l_1x_1 + l_2x_2 +$ &c., where l_1, l_2, &c., are proportional to the values of $\frac{dF}{dx_1}$, $\frac{dF}{dx_2}$, &c., at the point x_1', x_2', &c., and are such that $l_1^2 + l_2^2 + \ldots = 1$.

We have $d\xi_1 d\xi_2 \ldots d\xi_p = dx_1dx_2 \ldots dx_p$, and we take $d\xi_2 \ldots d\xi_p$ as the element of surface. We have then, putting dV for $dx_1dx_2 \ldots dx_p$, and dS for the element of surface

$$\int\left(\frac{du}{dx_1}\frac{dv}{dx_1} + \ldots\right)dV = \int v\frac{du}{d\xi_1}dS - \int v\left(\frac{d_2u}{dx_1^2} + \ldots\right)dV$$
$$= \int u\frac{dv}{d\xi_1}dS - \int u\left(\frac{d^2v}{dx_1^2} + \ldots\right)dV_1$$

We may, as before, apply this to finding the surface integral of the product of two harmonics $(a_1x_1 + a_2x_2 + \ldots)^n$, $(b_1x_1 + b_2x_2 + \ldots)_n$ over the surface $x_1^2 + x_2^2 + \ldots = \rho^2$.

The only difference in the general case is that the element of surface varies as r^{p-1} instead of as r^2. Unless $m = n$, the integral vanishes. If $m = n$, the result is

(remembering that $\frac{d}{d\rho}\int\int \ldots\ldots dx_1dx_2 \ldots dx_p = \frac{p\rho^{p-1}\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2}+1)}$),

$$(6) \quad \rho^{2n+p-1} \cdot \frac{\pi^{\frac{1}{2}n}\Gamma(n+1)}{2^n\Gamma(n+\frac{p}{2})}(a_1b_1 + a_2b_2 \ldots)^n.$$

Let us now find an expression for the rational integral harmonic, which is a function of x_p and $x_1^2 + x_2^2 + \ldots + x_p^2$.

If $f_1, f_2, .. f_{p-1}$ be real quantities such that the sum of their squares $=1$, then $(f_1x_1+f_2x_2+\ldots\ldots+f_{p-1}x_{p-1}+\iota x_p)^n$ is a harmonic, and if we take the sum of functions of this form for all real values of the f's such that the sum of their squares $=1$, we shall get what we want. This mode of derivation of the symmetrical harmonic is somewhat different from our former method, but it is easily seen to lead to the same result.

Consider first the integral

$$\iint\ldots(f_1x_1+\ldots\ldots+f_{p-1}x_{p-1}+\iota x_p)^n df_1\ldots\ldots df_{p-1}$$

with the integration extending to all real values of the f's for which the sum of their squares is not greater than ρ^2.

Change the variables f_1, f_2, &c., by a linear and orthogonal substitution to a new set, ϕ_1, ϕ_2, &c., of which

$$\sqrt{(x_1^2+\ldots+x_{p-1}^2)}\phi_1=f_1x_1+\ldots\ldots+f_{p-1}x_{p-1}$$

The integral is then

$$\iint\ldots(\sqrt{x_1^2+\ldots x_{p-1}^2}\phi_1+\iota x_p)^n d\phi_1 d\phi_2\ldots d\phi_{p-1}$$

and the limits are not changed.

Integrating with respect to $\phi_2\ldots\phi_{p-1}$ first, and disregarding numerical factors, we get

$$\int_{-\rho}^{+\rho}(\rho^2-\phi_1^2)^{\frac{p-2}{2}}(\sqrt{x_1^2+\ldots x_{p-1}^2}\phi_1+\iota x_p)^n d\phi_1.$$

Now differentiate this with respect to ρ, and in the result put $\rho=1$.

This evidently gives us what we started to find.

If we put $\phi_1=\cos\theta$, the result takes the form

$$\int_0^\pi(x_p+\iota\sqrt{x_1^2+x_2^2+\ldots x_{p-1}^2}\cos\theta)^n\sin^{p-3}\theta d\theta.$$

Now the surface integral of the product of this and a harmonic $(a_1x_1+\ldots+a_px_p)^n$ is obtained by substituting a_1 for x_1, a_2 for x_2, &c. in the above expression and multiplying by a numerical factor K.

This integral is $\therefore$ $\mathrm{K}(a_p)^n\int_0^\pi(1-\cos\theta)^n\sin^{p-3}\theta d\theta=\mathrm{M}(a_p)^n$, where M is a numeric, easily found.

By adding any number of harmonics of the typical form, we get a result which we may write

(7) $\int Q_n V_n dS = MV_n'$, *where* V_n' is the value of V_n at the pole of Q.

The surface value of the above symmetrical harmonic is

$$\int_0^\pi (x + \sqrt{x^2 - 1}\cos\theta)^n \sin^{p-3}\theta d\theta.$$

We may consider this as a function of the single variable x, and it is easy from the above results to show that

$$\int_{-1}^{+1} Q_n Q_m (1 - x^2)^{\frac{p-3}{2}} dx = 0$$

when m, n are different, but = a certain finite numerical quantity, easily found, when $m = n$.

The functions we have arrived at include, of course, as particular cases, simple and zonal Harmonic Functions. They are discussed at the end of Vol. I. of Heine's *Kugelfunctionen*.

The method of this paper is not mentioned in Heine, but I have found since I had it worked out that it is not new. In Professor Cayley's collected Works, Vol. I., page 397, will be found a short paper in which he proves (6) and (7). His proofs, however, are quite different from those I had arrived at, and have given above.

A Method of Teaching Electrostatics in School.

By J. T. Morrison, M.A., B.Sc.

The object of the paper was to suggest for the teaching of electrostatics a leading idea, which should *readily* co-ordinate all the facts, introduce no misleading inferences, and guide the course of learners in the direction of the most recent investigations—in all which respects the notion of attraction and repulsion is at least a partial failure. The leading idea or fact referred to is, that almost all electrostatic distributions, however complex, can be analysed into one or more repetitions of a certain simple system, which is called in the paper "an electrostatic system," and which may be described as follows :—Two equally and oppositely electrified conducting surfaces, facing each other, separated by any dielectric, and insulated from each other. A complete study of one system of this kind, and

of the very simple ways in which the establishment of one such system often necessitates the establishment of others, is therefore the fundamental study of electrostatics.

The following order may be adopted :—

After getting pupils to discover that there are two kinds of electrification, and to find out the difference between conductors and non-conductors, the investigation of an "electrostatic system" is begun.

A. *Electric Quantity and Distribution in "an electrostatic" system.*

Chief Result :—The surfaces are equally and oppositely electrified, and to each small part of one surface may be shown to correspond a small equally but oppositely electrified part of the other surface.

An important subsidiary principle may here be introduced —namely, the constancy of the quantity of electrification in a set of bodies that do not communicate with bodies that are outside the set. With the help of this principle, various distributions should be analysed into their constituent "electrostatic systems," and four important cases were so analysed in the paper.

B. *Mass-Motions in an "electrostatic system."*

Chief Result (*a*) :—There is a mass-acceleration of each part of the two surfaces, the direction of the mass-acceleration being perpendicular to the surface.

Definition of unit of electrification.

(*b*) Mass acceleration on surface carrying unit positive electrification $= -4\pi\sigma$.

C. *Energy and Potential Difference in an "electrostatic system."*

Definition of Potential. Equipotential Surfaces.

Chief Result :—Energy $= \frac{1}{2}$ Quantity $\times$ Potential Difference.

Seat and Nature of Electrostatic Energy.

Capacity, and Specific Inductive Capacity.

D. *Electric Discharge of a System.*

Electric Strength.

Electric Oscillations. Transformations undergone by the Electric Energy. Electrical Radiation.

There is no room to indicate the experimental means by which the method may be executed, nor to show how the various instruments may be introduced, their laws discovered, and constants calculated; but these matters will be fairly obvious.

Eighth Meeting, 13*th June* 1890.

R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

Note on the Orthomorphic Transformation of a circle into itself.

From a letter by Professor CAYLEY.

"The following is, of course, substantially well-known, but it strikes me as rather pretty :—to find the Orthomorphic transformation of the circle

$$x^2 + y^2 - 1 = 0$$

into itself. Assume this to be

$$x_1 + \iota y_1 \frac{A(x + \iota y) + B}{1 + C(x + \iota y)}.$$

Then, writing A′, B′, C′ for the conjugates of A, B, C, we have

$$x_1 - \iota y = \frac{A'(x - \iota y) + B'}{1 + C'(x - \iota y)};$$

and then

$$x_1^2 + y_1^2 = \frac{AA'(x^2 + y^2) + AB'(x + \iota y) + A'B(x - \iota y) + BB'}{1 + C(x + \iota y) + C'(x - \iota y) + CC'(x^2 + y^2)},$$

which should be an identity for $x^2 + y^2 = 1$, $x_1^2 + y_1^2 = 1$.

Evidently $C = AB'$, whence $C' = A'B$, and the equation then is

$$1 + AA'BB' = AA' + BB',$$

i.e.,

$$(1 - AA')(1 - BB') = 0.$$

But $BB' = 1$ gives the illusory result

$$x_1 + \iota y_1 = B,$$

therefore

$$1 - AA' = 0,$$

and the required solution thus is

$$x_1 + \iota y_1 = \frac{A(x + \iota y) + B}{1 + AB'(x + \iota y)};$$

where A is a unit-vector (say $A = \cos\lambda + \iota\sin\lambda$) and B, B′ are conjugate vectors. Or, writing $B = b + \iota\beta$, $B' = b - \iota\beta$, the constants are λ, b, β; 3 constants as it should be."

Quaternion Synopsis of Hertz' View of the Electrodynamical Equations.

By Professor TAIT.

Note on Menelaus's Theorem.

By R. E. ALLARDICE, M.A.

§ 1. The object of this note is, in the first place, to show that Menelaus's Theorem, regarding the segments into which the sides of a triangle are divided by any transversal, is a particular form of the condition, in trilinear co-ordinates, for the collinearity of three points; and, in the second place, to point out an analogue of Menelaus's Theorem in space of three dimensions.

§ 2. In the usual system of areal co-ordinates, the x-co-ordinate of P (fig. 52) is $\Delta PBC/\Delta ABC$, that is PD/AD. Now let D, E, F, be three points in BC, CA, AB, respectively, dividing these sides in the ratios l_1/m_1, l_2/m_2, l_3/m_3; then the co-ordinates of D, E, F, are proportional to $(0, m_1, l_1)$, $(l_2, 0, m_2)$, $(m_3, l_3, 0)$. Hence the condition that D, E, F, lie on the straight line $Ax + By + Cz = 0$ is

$$\begin{vmatrix} 0 & m_1 & l_1 \\ l_2 & 0 & m_2 \\ m_3 & l_3 & 0 \end{vmatrix} = 0,$$

that is, $l_1 l_2 l_3 + m_1 m_2 m_3 = 0$, which is Menelaus's Theorem.

§ 3. In space of three dimensions we may use the corresponding system of tetrahedral co-ordinates, and obtain a theorem analogous to that of Menelaus.

Let BCD (fig. 53) be one of the faces of the tetrahedron; and put $a_2 = PB'/BB' = \Delta PCD/BCD$, $a_3 = PC'/CC' = \Delta PDB/\Delta CDB$, etc. Then the co-ordinates of P, Q, R, S, points in the four faces of the tetrahedron, may be written $(0, a_2, a_3, a_4)$, $(b_1, 0, b_3, b_4)$, etc.; and the condition that these four points be coplanar is

$$\begin{vmatrix} 0 & a_2 & a_3 & a^4 \\ b_1 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} = 0,$$

where a_2, a_3, and a_4 may be taken to be the three areas into which the point P divides the face BCD; and this condition is the analogue of Menelaus's Theorem for space of three dimensions.

Historical notes on a geometrical problem and theorem.

By J. S. MACKAY. M.A., LL.D.

The problem is

Between two sides of a triangle to inflect a straight line which shall be equal to each of the segments of the sides between it and the base.

This problem was brought before the Society at the January meeting in 1884, and a solution of it by Mr James Edward will be found in our *Proceedings*, Vol. II, pp. 5–6, a second by myself in Vol. II., p. 27 (10th April 1884), a third by Mr R. J. Dallas in Vol. III., pp. 41–2 (9th January 1885). Solutions of a slightly more general problem were also given by myself in Vol. III., pp. 40–1, and reference made to the *Educational Times*, Vol. 37, p. 328 (1st October 1884), and to Vuibert's *Journal de Mathématiques Élémentaires*, 9e année, p. 45 (15th December 1884).

I have since found that the more general problem was proposed by Monsieur J. Neuberg in the *Nouvelle Correspondance Mathématique*, Vol, I., p. 110 (1874–5), and solved by him in Vol. II, p. 248 (1876); and quite recently I have discovered the first problem to go as far back as 1773–4. Here is how it occurs.

In the *Ladies' Diary* for 1773, Mr Thomas Moss proposes for solution the following:—

*The difference of the sides including a known angle of a plane triangle being given, and also the sum of one of those sides and that opposite the given angle, to construct the triangle.**

In 1774 the question is thus answered by Mr John Turner:—

"*Analysis.* Suppose the thing done, and that ABC is the

---

* Thomas Leybourn's *Mathematical Questions proposed in the Ladies' Diary*, Vol. II., p. 377 (1817).

triangle, of which are given the angle A, the sum AC + CB, and the difference AB − AC. Then also will be given AB + BC, it being evidently equal to AC + CB + the given difference AB − AC; and therefore if AD and AE be made respectively equal to the given sums AB + BC and AC + CB, the triangle ADE will also be given; and then between the sides of this given triangle we have only to apply BC so as to cut off BD and CE each equal to it; whence appears this easy

"*Construction.* Form (Fig. 44) the given angle DAE and make DA and AE equal to the two given sums of each side and the base; draw any line FG parallel to DE, and make EH and HI each equal to DF; then draw EIB, and lastly BC parallel to HI cutting off the triangle ABC required."

The demonstration is omitted as unnecessary.

The theorem is

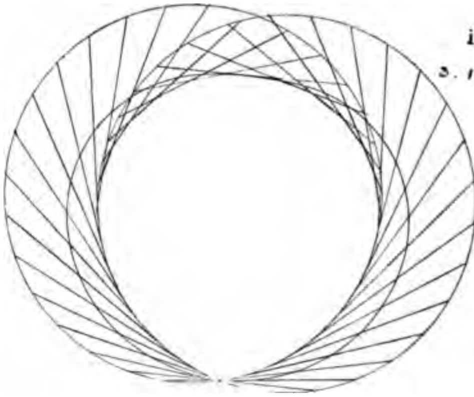
The middle points of the three diagonals of a complete quadrilateral are collinear.

It was Carnot (*Essai sur la théorie des transversales*, 1806, § 6) who bestowed the name complete quadrilateral on a system of four straight lines no three of which are concurrent, but he does not seem to have been aware of this property of the three diagonals. In Cremona's *Elements of Projective Geometry* the property is ascribed to Gauss (Collected Works, Vol. IV., p. 391), and Dr Baltzer says, "Diese Eigenschaft des Vierecks ist von Gauss 1810 bemerkt worden (v. Zach monatl. Korresp. 22, p. 115)."

In the *Ladies' Diary* for 1795 the following question is proposed by Mr J. T. Connor, Lewes Academy, and in 1796 two solutions of it are given, the first by Major Henry Haldane of the Royal Engineers, and the second by Mr John Ryley of Leeds.

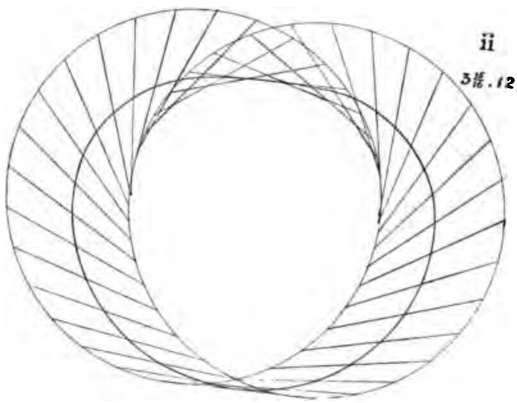
In a triangle ABC, drawing any two lines AE, BD from the extremities of one side, to terminate in the other two sides; and thereby form a trapezium CDKE; if the diagonals DE and CK of that trapezium be bisected in the points M and N, and the first side of the triangle in the point P, these three points P, M, N will be in a straight line. Required the demonstration." *

* Thomas Leybourn's *Mathematical Questions proposed in the Ladies' Diary*, Vol. III., p. 291 (1817).



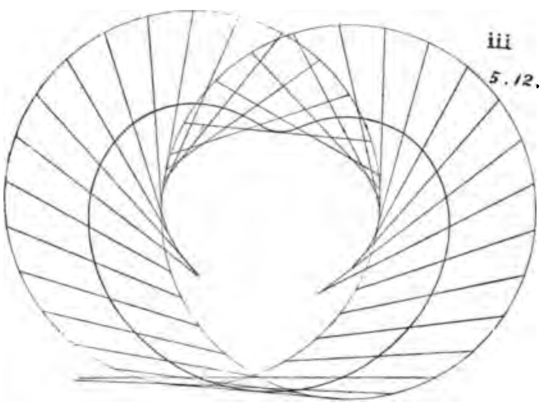
i

3 . 12, 6



ii

$3\frac{15}{16}$. 12, $5\frac{1}{4}$

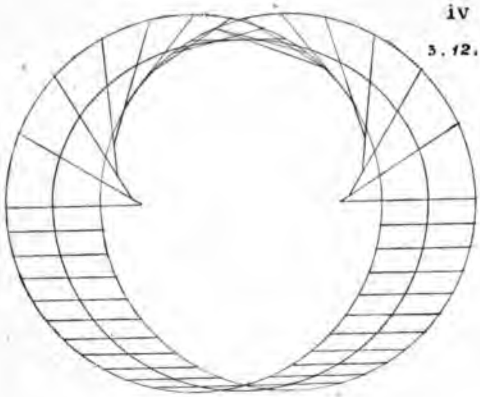


iii

5 . 12, 6

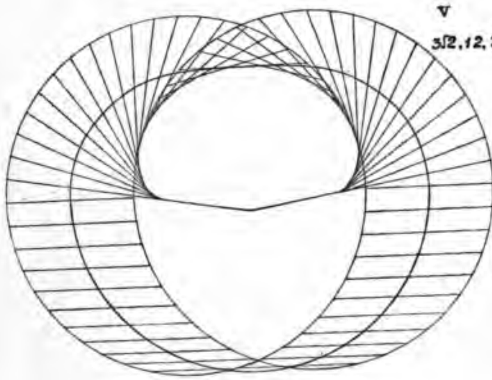
iv

$3, 12, 3$



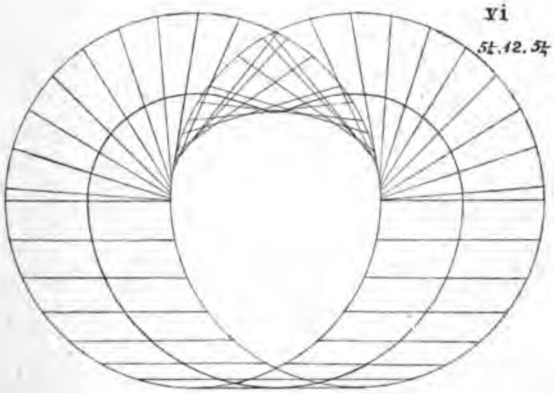
v

$3\sqrt{2}, 12, 3\sqrt{2}$



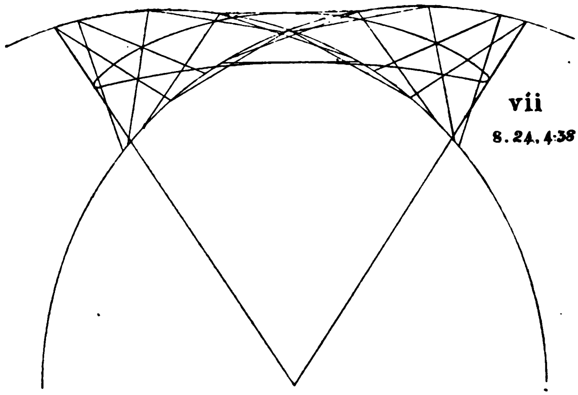
vi

$5\frac{1}{4}, 12, 5\frac{1}{4}$



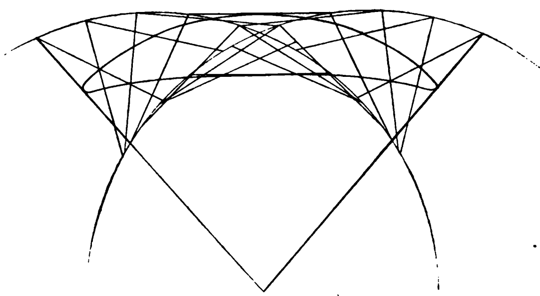
vii

8. 24, 4·38



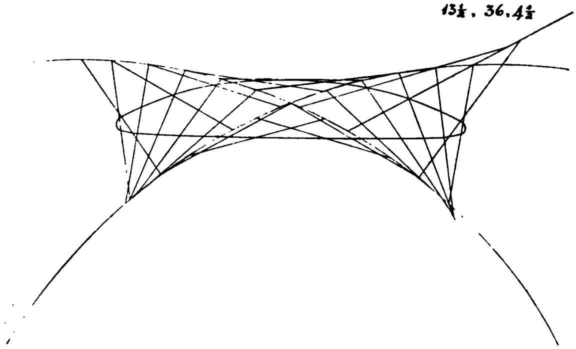
viii

6·84, 18, 4·5



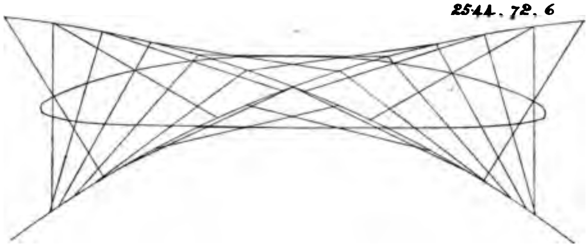
ix

$13\frac{1}{2}$, 36, $4\frac{1}{2}$



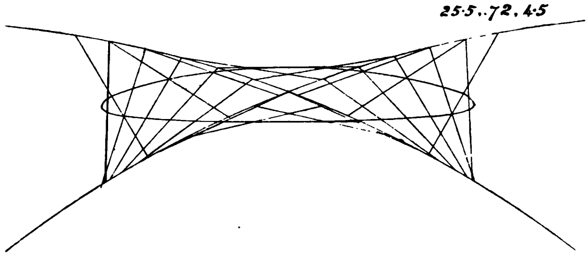
x

25·44 . 72 . 6



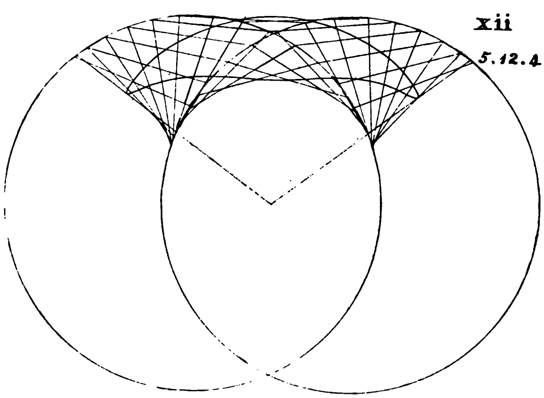
xi

25·5 . . 72 . 4·5



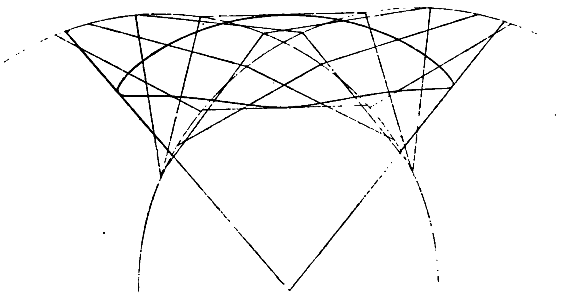
xii

5 . 12 . 4



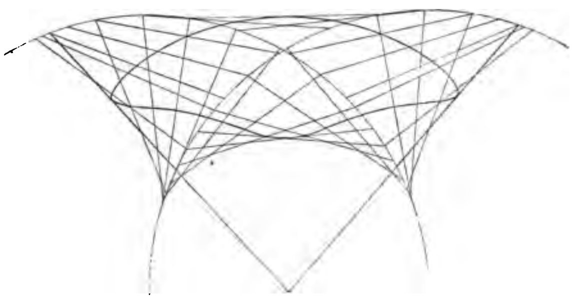
xiii

$8\frac{1}{4}$. 18 . $5\frac{1}{4}$



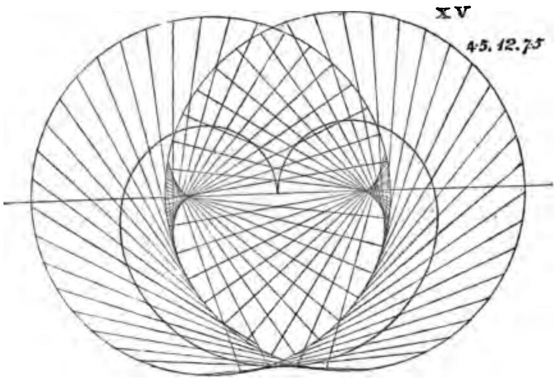
xiv

9 . 18 . 6



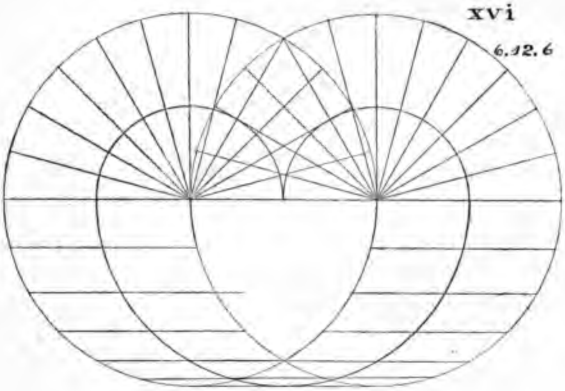
xv

4·5. 12. 7·5



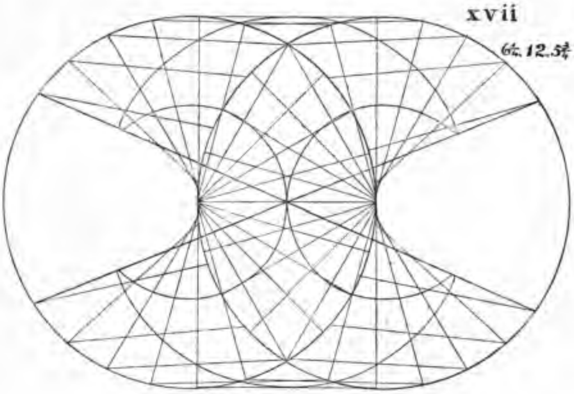
xvi

6.12.6



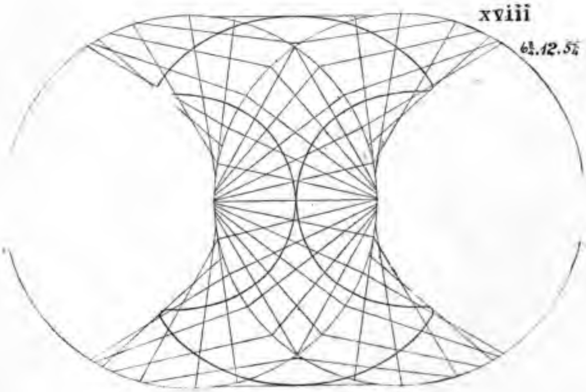
xvii

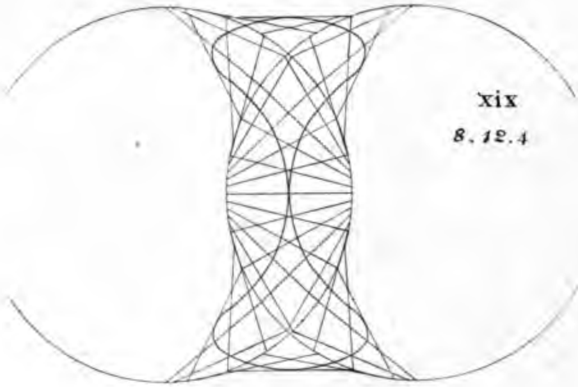
$6\frac{1}{4}.12.5\frac{3}{4}$



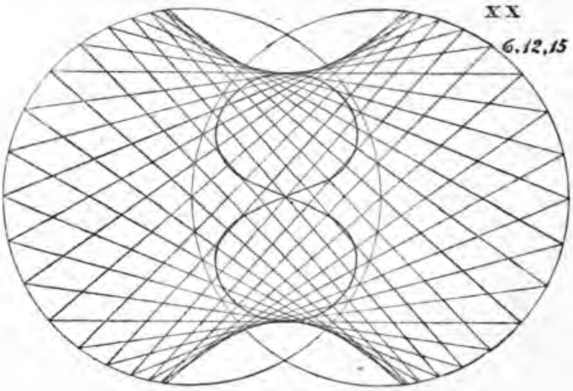
xviii

$6\frac{1}{4}.12.5\frac{3}{4}$

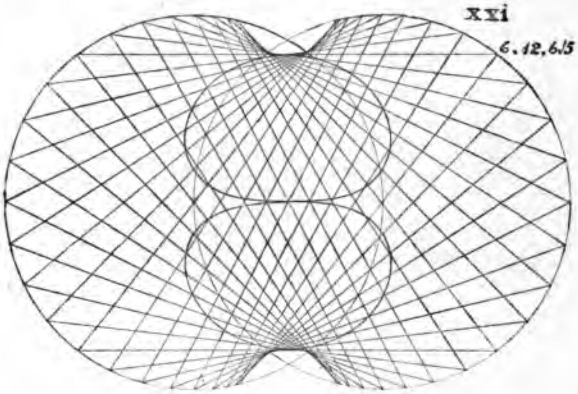




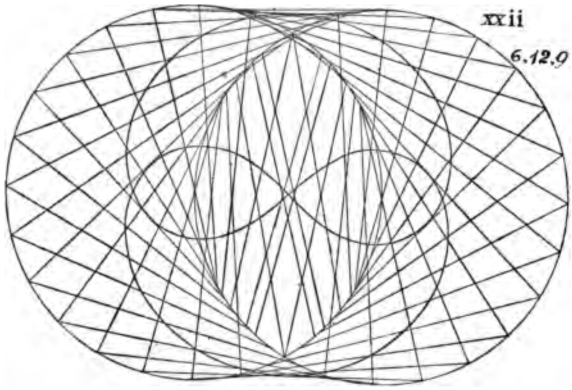
xix
8.12.4



xx
6.12,15

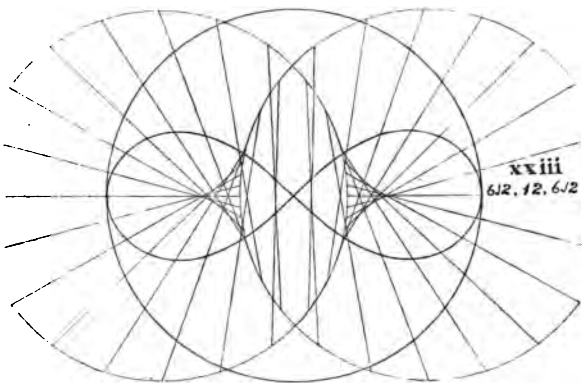


xxi
6.12,6.15



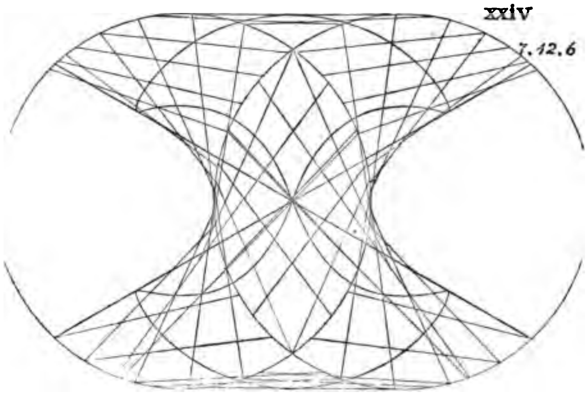
xxii

6.12.9



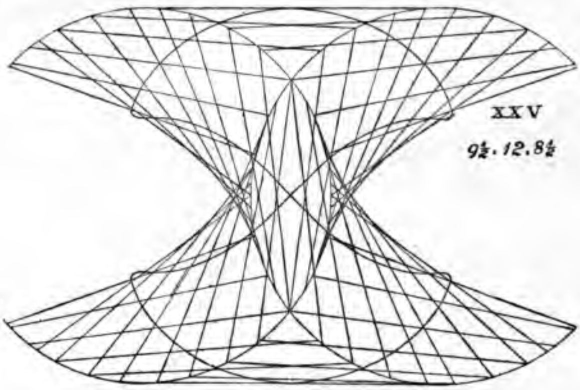
xxiii

$6\sqrt{2}$. 12. $6\sqrt{2}$



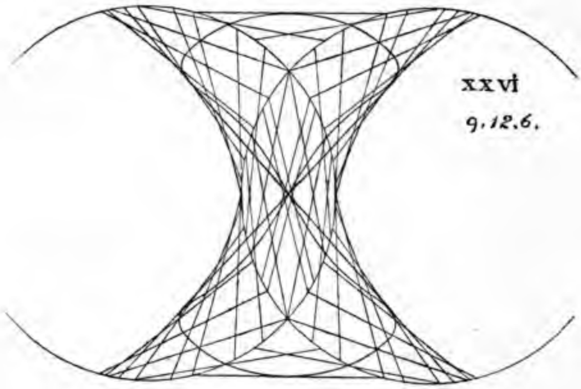
xxiv

7.12.6



XXV

$9\frac{1}{2}.12.8\frac{1}{2}$

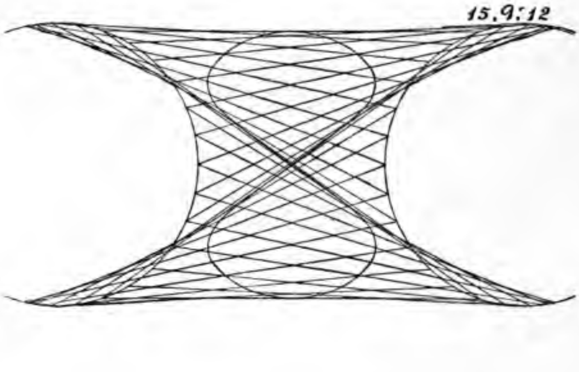


XXVI

9.12.6.

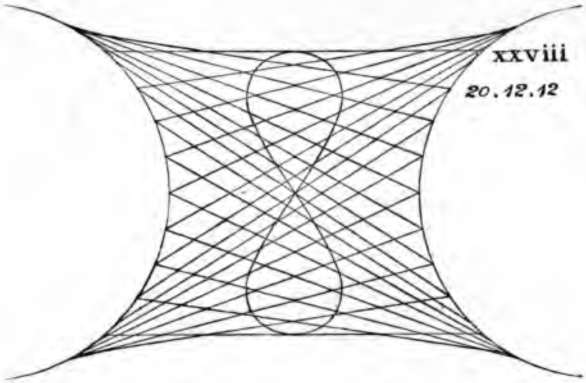
XXVII

15.9:12

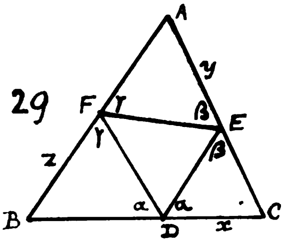


xxviii

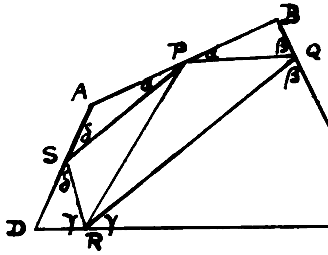
20.12.12



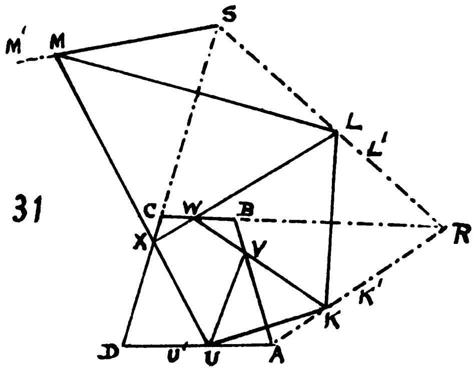
29



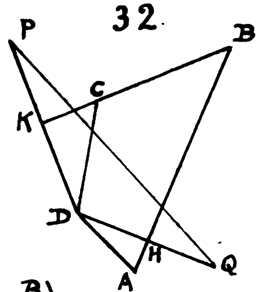
30



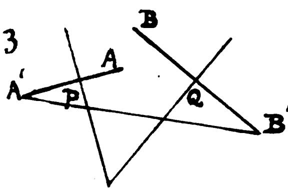
31



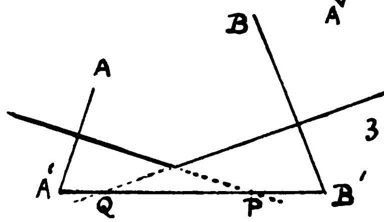
32



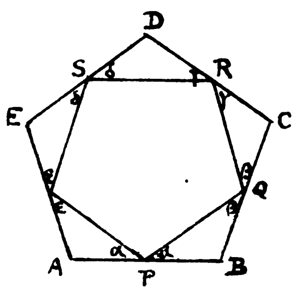
33



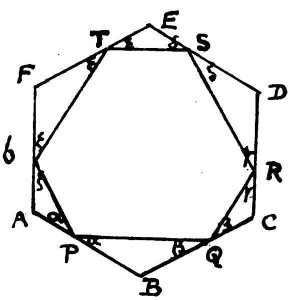
34



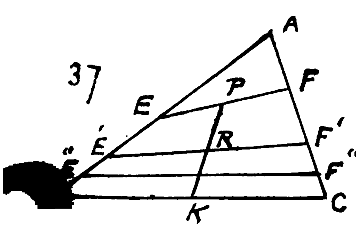
35



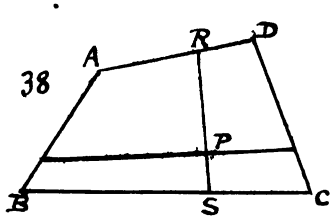
36



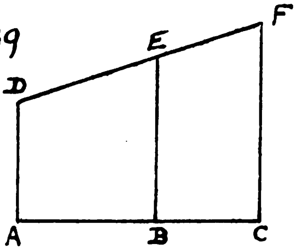
37



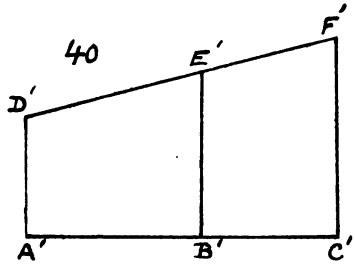
38



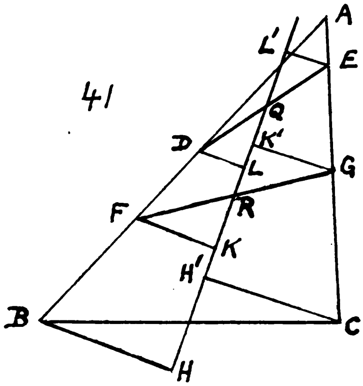
39



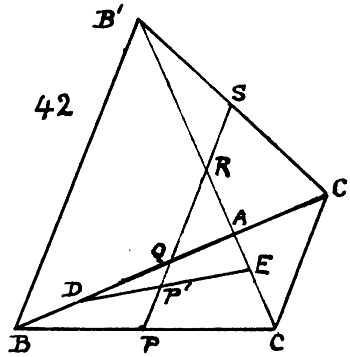
40



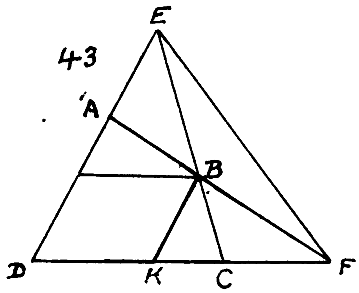
41



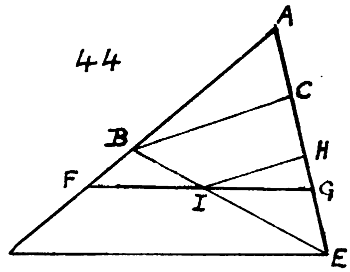
42



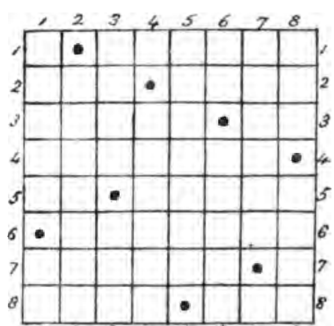
43



44



45



46

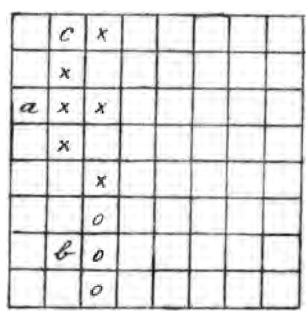


Fig. 47

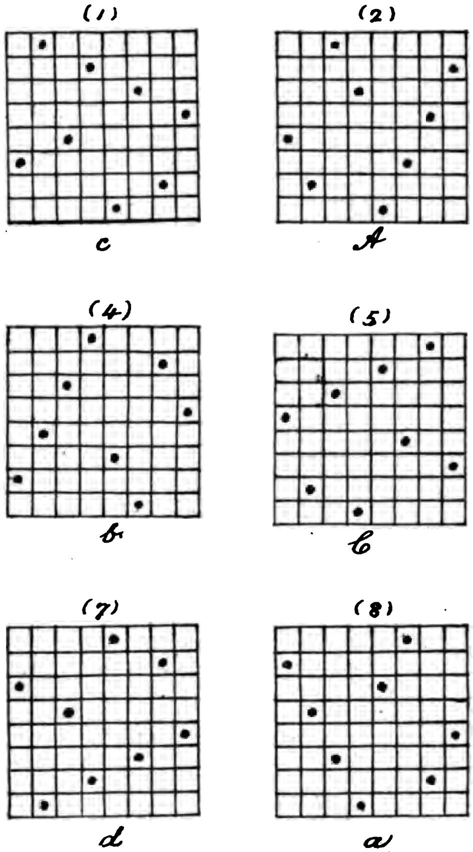


Fig. 49

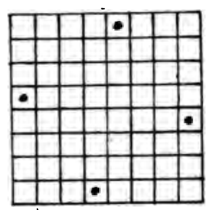


Fig. 48

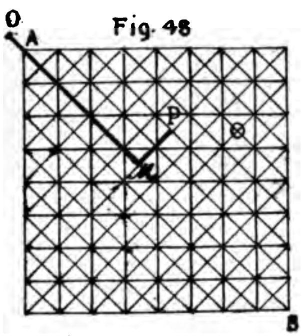


Fig. 54

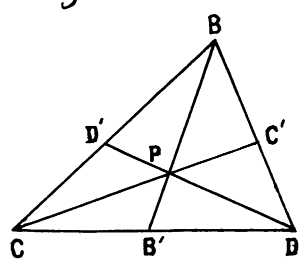


Fig. 50

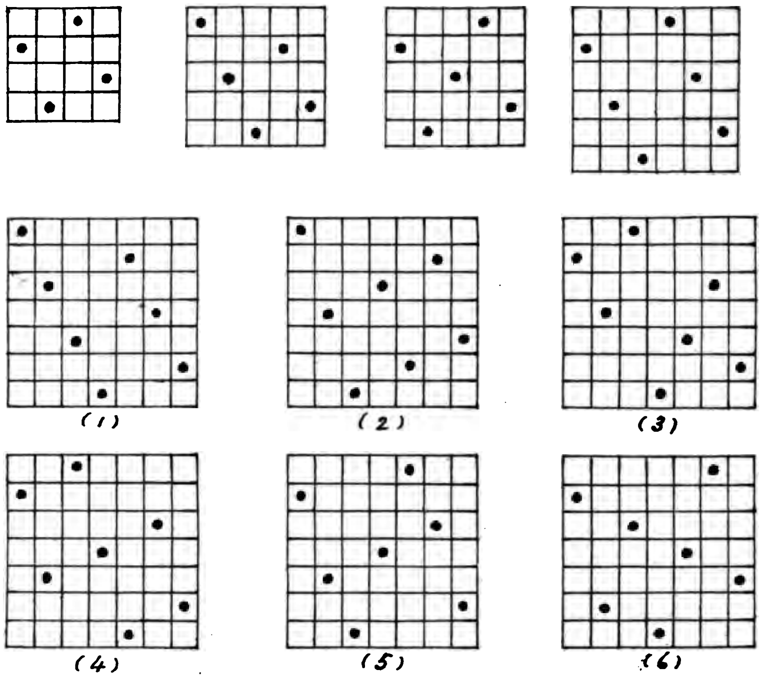
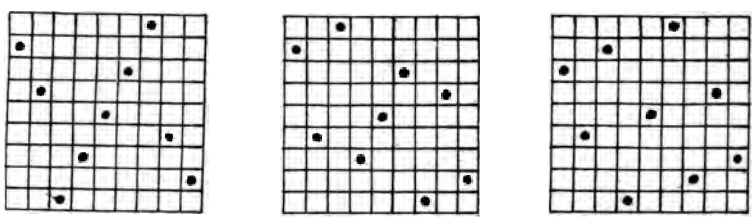


Fig. 51



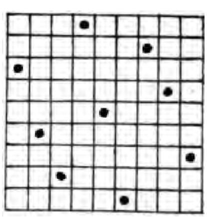


Fig. 52

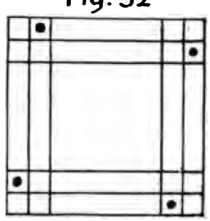
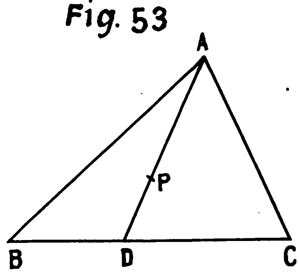
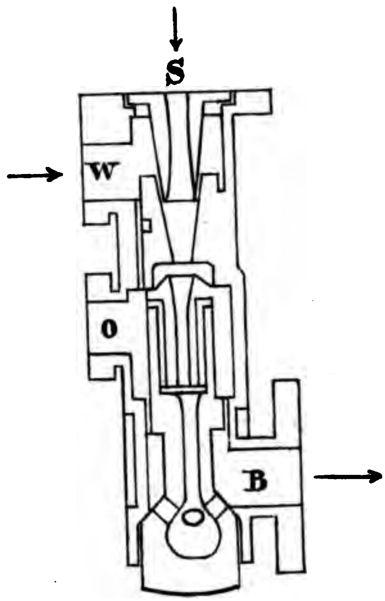


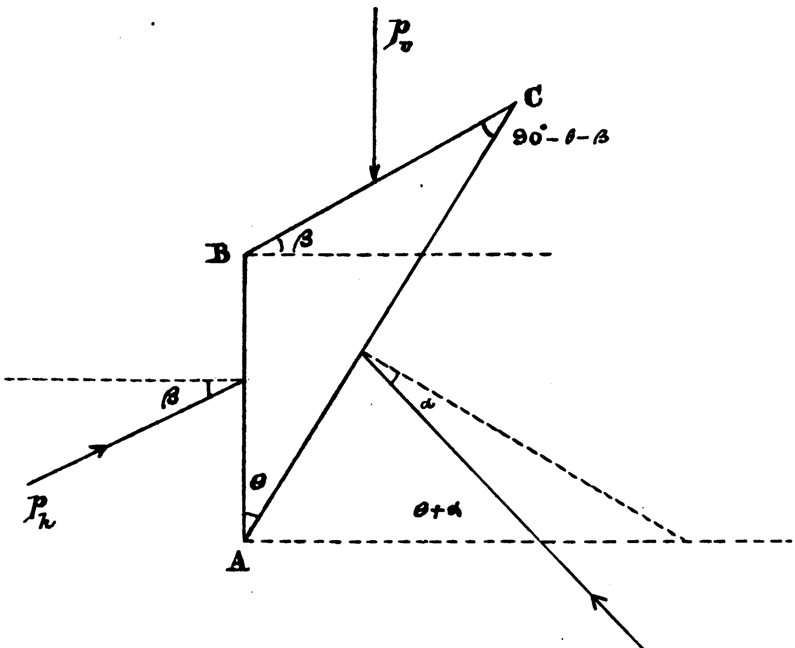
Fig. 53





S
W
O
B

Fig. 55



p_v
C
$90° - \theta - \beta$
B
β
β
α
θ
$\theta + \alpha$
p_h
A
R

Fig. 56

STORAGE